Subsidiary
Mathematics

for Rwanda Secondary Schools

Learner's Book 4

Writing Team
Ngezahayo Emmanuel
Icyingeneye Pacifique

FOUNTAIN PUBLISHERS
www.fountainpublishers.co.ug

Fountain Publishers Rwanda Ltd
P.O. Box 6567
Kigali, Rwanda
E-mail: fountainpublishers.rwanda@gmail.com
sales@fountainpublishers.co.ug;
publishing@fountainpublishers.co.ug
Website: www.fountainpublishers.co.ug

First printed 2016

ISBN: 978-9970-19-365-3

Contents

Introduction

Changes in schools

This text book is part of the reform of the school curriculum in Rwanda: that is changes in what is taught in schools and how it is taught. It is hoped this will make what you learn in school useful to you when you leave school, whatever you do then.

In the past, the main thing in schooling has been to learn knowledge – that is facts and ideas about each subject. Now the main idea is that you should be able to use the knowledge you learn by developing skills or competencies. These skills or competencies include the ability to think for yourself, to be able to communicate with others and explain what you have learnt, and to be creative, that is developing your own ideas, not just following those of the teacher and the text book. You should also be able to find out information and ideas for yourself, rather than just relying on what the teacher or text book tells you.

Activity-based learning

This means that this book has a variety of activities for you to do, as well as information for you to read. These activities present you with material or things to do which will help you to learn things and find out things for yourself. You already have a lot of knowledge and ideas based on the experiences you have had and your life within your own community. Some of the activities, therefore, ask you to think about the knowledge and ideas you already have.

In using this book, therefore, it is essential that you do all the activities. You will not learn properly unless you do these activities. They are the most important part of the book.

In some ways this makes learning more of a challenge. It is more difficult to think for yourself than to copy what the teacher tells you. But if you take up this challenge you will become a better person and become more successful in your life.

Group work

You can also learn a lot from other people in your class. If you have a problem it can often be solved by discussing it with others. Many of the activities in the book, therefore, involve discussion or other activities in groups or pairs. Your teacher will help to organise these groups and may arrange the classroom so you are always sitting in groups facing each other. You cannot discuss properly unless you are facing each other.

Research

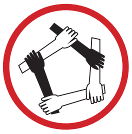
One of the objectives of the new curriculum is to help you find things out for yourself. Some activities, therefore, ask you to do research using books in the library, the internet if your school has this, or other sources such as newspapers and magazines. This means you will develop the skills of learning for yourself when you leave school. Your teacher will help you if your school does not have a good library or internet.

Icons

To guide you, each activity in the book is marked by a symbol or icon to show you what kind of activity it is. The icons are as follows:

Practical Activity icon

The hand indicates a practical activity such as curve sketching, draw figures, to have a selection of objects, individually or in a group and then present your results or comments.



Group Work icon

Group work means that you are expected to discuss something in groups and report back on what your group discussed. In this way you learn from each other and how to work together as a group to address or solve a problem.

Pairing Activity icon

This means that you are required to do the activity in pair, exchange ideas and write down your results.

Research Activity icon

Some activities require you to do research either by reading textbooks or using the internet.

Good luck in using the book.

Unit 1 Fundamentals of Trigonometry

My goals

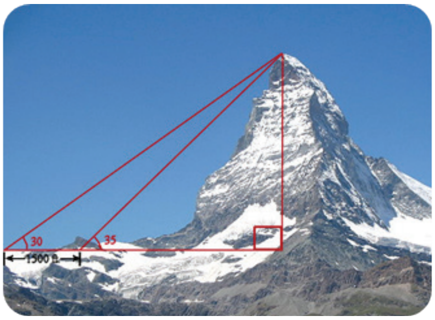
By the end of this unit, i will explain:

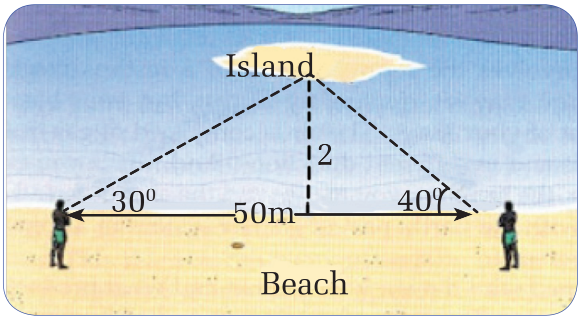
- Trigonometric concepts.
- Triangle and applications.

Intorduction

Trigonometry studies relationship involving lengths and angle of a triangle.

The techniques in trigonometry are used for finding relevance in navigation particularly satellite systems and astronomy, naval and aviation industries, oceanography, land surveying, and in cartography (creation of maps). Now those are the scientific applications of the concepts in trigonometry, but most of the mathematics we study would seem (on the surface) to have little real-life application. Trigonometry is really relevant in our day to day activities. The following pictures show us the areas where this science finds use in our daily activities. In this unit we will see how we can use this to resolve problems we might encounter.





Required outcomes

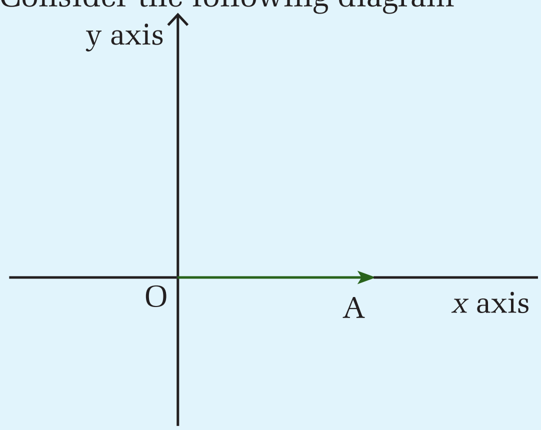
After completing this unit, the learners should be able to:

- Define sine, cosine, and tangent (cosecant, secant and cotangent) of any angle – know special values.
- Convert radians to degree and vice versa.
- Use trigonometric identities.
- Apply trigonometric formulae in real world problems.

1. Trigonometric concepts

Activity 1

Consider the following diagram



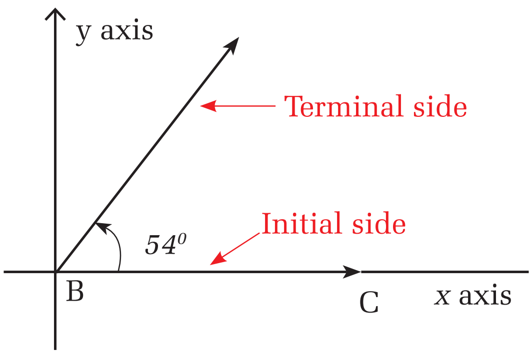
Copy the diagram and rotate vector $\overrightarrow{OA}$ by an angle of

2. 30 degrees
3. -45 degrees
4. 120 degrees

Trigonometry is the study of how the sides and angles of a triangle are related to each other. A rotation angle is formed by rotating an **initial side** through an angle, about a fixed point called **vertex**, to terminal position called **terminal side**.

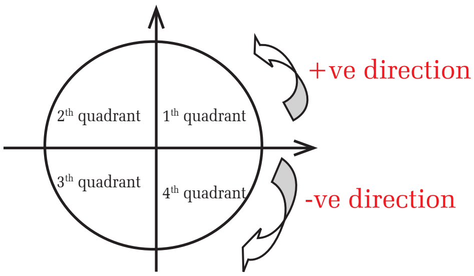
It is drawn in what is called standard position if the initial side is on the positive *x-axis and* the vertex of the angle is at the origin.

Example 1



Angles in standard position that have a common terminal side are called **co-terminal angles**; the measure of smallest positive rotation angle is called **principal angle.** Angle is positive if rotated in a counterclockwise direction and negative when rotated clockwise.

Angles are named according to where their terminal side lies, for instance, the *x*-axis and *y*-axis divide a plane into four quadrants as follow



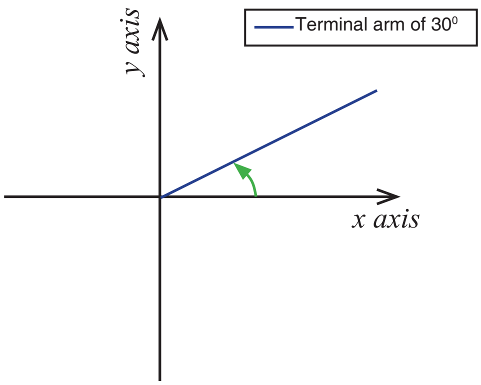
Example 2

Draw each of the following angles in standard position and which of these angles is co-terminal to 30^0?

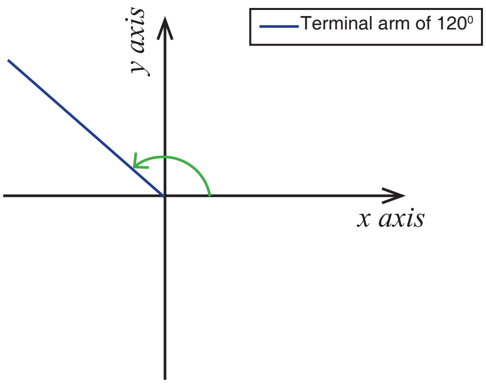
a) 30^0 b) 120^0 c) -230^0 d) 750^0 e) -330^0

Solution

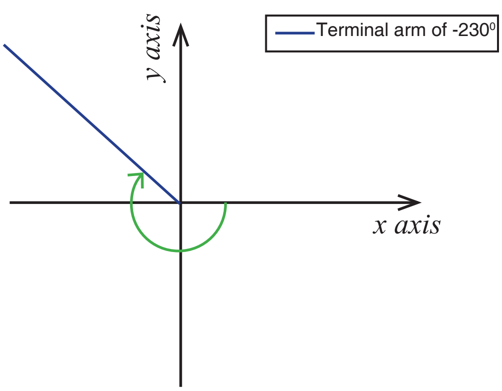
a)



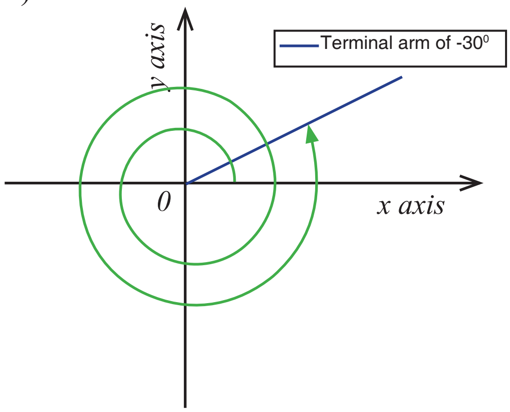
b)



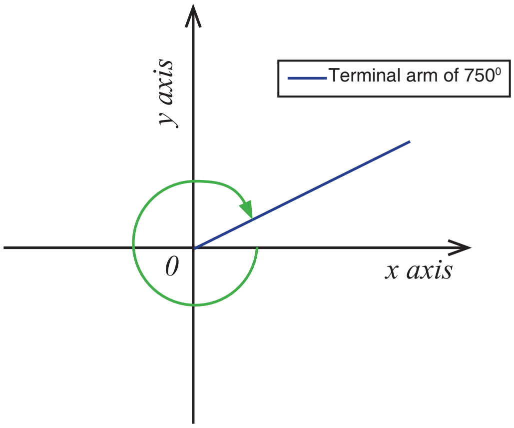
c)



d)



e)



750^0 **and** -330^0 **are co-terminal to** 30^0

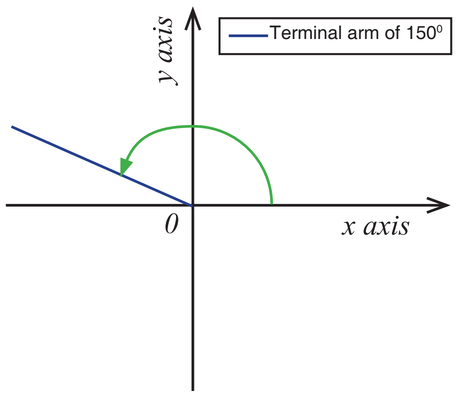
Example 3

Draw each of the following angles in standard position and indicate in which quadrant the terminal side is.

a) 150° b) -35° c) 210°

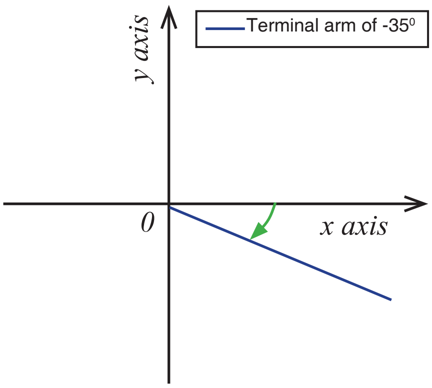
Solution

a)



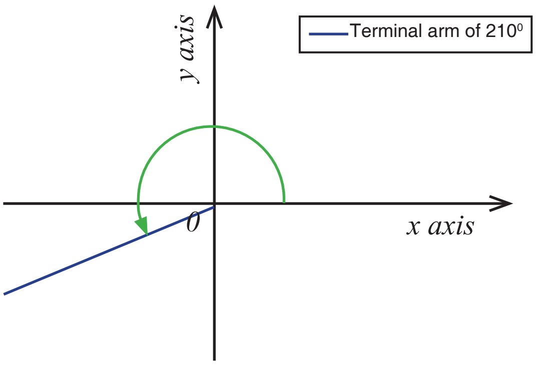
150^0 lies in 2^{nd} quadrant.

b)



-35^0 lies in 4^{th} quadrant

c)



210^0 lies in 3^{rd} quadrant.

Exercise 1

1. Draw each of the following angles in standard position and which of these angles is co-terminal to 20^0?
   a) 20^0 b) -200^0 c) 740^0 d) -340^0
2. Draw each of the following angles in standard position and indicate in which quadrant the terminal side is.
   a) 40^0 b) -235^0 c) 280^0

Measure of an angle

Activity 2

If $\frac{x \text{ degrees}}{180} = \frac{y}{\pi} = \frac{z}{200}$ find

1. z for 24 degrees
2. z for 248 degrees
3. y for 180 degrees
4. y for 270 degrees

The amount we rotate the angle is called the measure of the angle and is measured in following units:

a) Sexagesimal system

Unit is degree (written with a superscript∘). One degree (1^0) is $\frac{1}{90}$ of the right angle. In angular measure, the degree is subdivided into minutes and seconds ($'$ and $''$). $1^0 = 60'$, $1' = 60''$.

Example 4

The angle which measures 12 degrees, 35 minutes and 15 seconds will be denoted by $12^0 35' 15''$.

Degrees (º), minutes (‘), seconds (‘’) to decimal degrees and vice versa

Let d represents the integer degrees, dd represents the decimal degree, m represents minutes and s represents seconds. Then:

$$d = integer(dd) \qquad m = integer((dd - d)\times 60)$$

$$s = \left(dd - d - \frac{m}{60}\right) \times 3600$$

For an angle with d integer degrees m minutes and s seconds ($d^0m's''$), the decimal degree (dd) is:

$$dd = d + \frac{m}{60} + \frac{s}{360}$$

Example 5

Convert 30.263888889º to $d^0m's''$ ($d^0m's''$ notation)

Solution

$$d = integer(30.263888889^\circ) = 30^\circ$$

$$m = integer((dd - d)\times 60) = integer((30.263888889^\circ - 30^\circ)\times 60) = 15'$$

$$s = \left(dd - d - \frac{m}{60}\right)\times 3600 = \left(30.263888889^\circ - 30^\circ - \frac{15}{60}\right)\times 3600 = 50''$$

so, 30.263888889º = 30º 15' 50"

Example 6

Convert 30 degrees 15 minutes and 50 seconds angle to decimal degrees

Solution

30º 15’ 50”

The decimal degrees dd is equal to:

$$dd = d + \frac{m}{60} + \frac{s}{60}$$

$$= 30 + \frac{15}{60} + \frac{50}{3600}$$

$$= 30.263888889^\circ$$

b) Centesimal system

Unit is grade. One grade is equal to $\frac{1}{100}$ of the right angle and is subdivided into:

decigrade: $\frac{1}{10}$ grades,

centigrade: $\frac{1}{100}$ grades and

milligrade: $\frac{1}{1000}$ grades

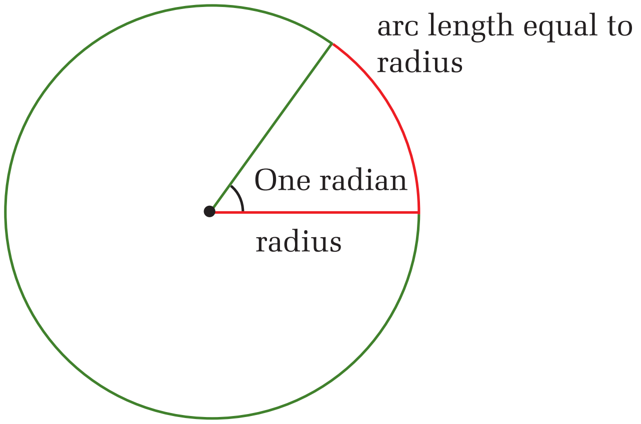
Example 7

An angle which measures 82 grades, 7 decigrades, 2 centigrades and 5 milligrades will be denoted by $82^{G},725$

c) Radian

A central angle of a circle is an angle with a vertex at the centre of a circle. An intercepted arc is the portion of the circle with endpoints on the sides of the central angle and remaining points within the interior of the angle.

When a central angle intercepts an arc that has the same length as a radius of the circle, the measure of the angle is defined to be **one radian**.



Like degrees, radian measures the amount of the rotation from the initial side to the terminal side of an angle.

Proportions between three units

Because the circumference of the circle is $2\pi r$, there are 2π radians in any circle (as radius must be equal to the arc intercepted by the angle).

Since 2π radians $= 360$ degrees then π radians $= 180$ degrees

To convert between degrees and radians, we can use the proportion

$$\frac{D}{180} = \frac{R}{\pi},$$

where D stands for degrees and R stands for radians.

Also $1G = \frac{1}{100} \times 90^\circ$ or $100G = 90^\circ$ or $200G = 180^\circ$

To convert between degrees and grades, we can use the proportion $\frac{D}{180} = \frac{G}{200}$

where D stands for degrees and G stands for grades. The combined relation is

$$\frac{D}{180} = \frac{R}{\pi} = \frac{G}{200}$$

where D stands for degree, R for radians, G for grades and $\pi = 3.14...$

This relation can be split into 3 relations:

$\boxed{\frac{D}{180} = \frac{R}{\pi}}$, $\boxed{\frac{D}{180} = \frac{G}{200}}$ and $\boxed{\frac{R}{\pi} = \frac{G}{200}}$

Example 8

Convert 90^0 to radians and grades;

$$\frac{D}{180} = \frac{R}{\pi} \Leftrightarrow \frac{90}{180} = \frac{R}{\pi}$$

$$\Leftrightarrow R = \frac{90\pi}{180} = \frac{\pi}{2} \text{ or } R = 1.57$$

$$\frac{D}{180} = \frac{G}{200} \Leftrightarrow \frac{90}{180} = \frac{G}{200}$$

$$\Leftrightarrow G = \frac{90 \times 200}{180} = 100$$

Thus, $90^0 = \frac{\pi}{2} \textit{radians} \text{ or } 1.57 \textit{ radians}$ and

$90^0 = 100 \textit{ grades}$

Example 9

Convert 20 grades to radians and degrees;

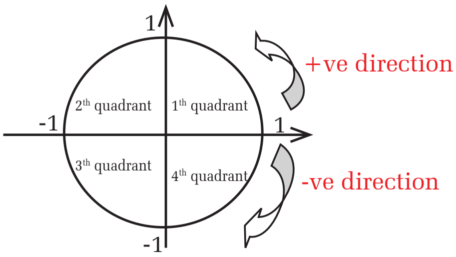
$$\frac{R}{\pi}=\frac{G}{200}\Leftrightarrow\frac{R}{\pi}=\frac{20}{200} \qquad \frac{D}{180}=\frac{G}{200}\Leftrightarrow\frac{D}{180}=\frac{20}{200}$$

$$\Leftrightarrow R=\frac{20\pi}{200}=\frac{\pi}{10}\ \ or\ R=0.314 \qquad \Leftrightarrow D=\frac{20\times180}{200}=18$$

Thus, $20\,grades=18\deg$ and

$$20\,grades=\frac{\pi}{10}radians\,or\,0.314\,radians$$

Notice: Unit circle



A **unit circle** is a circle of radius one centered at the origin (0,0) in the Cartesian coordinate system in the Euclidean plane.
In the unit circle, the coordinate axes delimit four quadrants that are numbered in an anticlockwise direction. Each quadrant measures 90 degrees, meaning that the entire circle measures 360 degrees or 2π radians.

Exercise 2

1. Convert 220 grades to radians and degrees
2. Convert 1240 degrees to radians and grades
3. Convert $\frac{2}{5}\pi$ to degrees
4. Convert 5.6^0 to $d^0m's''$ system

Trigonometric ratios of acute angles

Activity 3

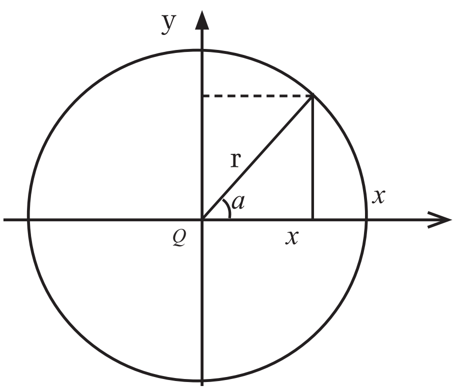
Construct two right angled triangles, one of which is an enlargement of the other.

How is the side opposite to the right angle (or the longest side) called?
For both triangles, consider an angle and compute the following ratios.

- Opposite side to the considered angle and hypotenuse.
- Adjacent side and hypotenuse.
- Opposite side to the considered angle and adjacent side.

How can you conclude?

Consider the following circle with radius r.



In this triangle, we define the following six ratios:

- **The** three **primary trigonometric values**
- The ratio $\frac{x}{r}$ is called cosine of the angle α, noted $\cos\alpha$.
- The ratio $\frac{y}{r}$ is called sine of the angle α, noted $\sin\alpha$.
- The ratio $\frac{y}{x}$ is called tangent of the angle α, noted $\tan\alpha$.

- **The** three **reciprocal trigonometric values**
  - **Secant** of the angle α, denoted $\sec\alpha$ is the ratio $\frac{r}{x}$,
  - **Cosecant** of the angle α, denoted $\csc\alpha$ is the ratio $\frac{r}{y}$ and
  - **Cotangent** of the angle α, denoted $\cot\alpha$ is the ratio $\frac{x}{y}$.

Observing the triangle in the above circle, we see that r is the hypotenuse, x is the adjacent side and y is the opposite side.

Then, $\sin\alpha = \frac{opposite\ side}{hypotenuse}$, $\cos\alpha = \frac{adjacent\ side}{hypotenuse}$, $\tan\alpha = \frac{opposite}{adjascent}$.

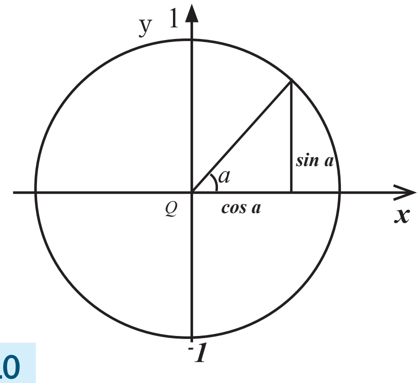
$$\csc\alpha = \frac{hypotenuse}{opposite} = \frac{1}{\sin\alpha}, \sec\alpha = \frac{hypotenuse}{adjascent} = \frac{1}{\cos\alpha} \text{ and } \cot\alpha = \frac{adjacent}{opposite} = \frac{1}{\tan\alpha}.$$

If the circle is unit, then;

$$\cos\alpha = adjacent\ side = \frac{x - coordinate}{1} \Rightarrow |\cos\alpha| \le 1,$$

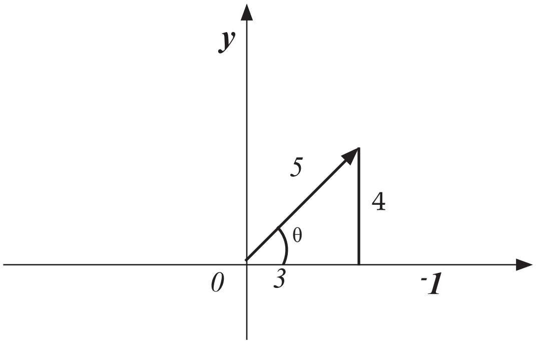
$$\sin\alpha = opposite\ side = \frac{y - coordinate}{1} \Rightarrow |\sin\alpha| \le 1 \text{ and}$$

$$\tan\alpha = \frac{opposite\ side}{adjacent\ side} = \frac{y - coordinate}{x - coordinate}.$$



Example 10

Calculate the six trigonometric values for the diagram.



Solution

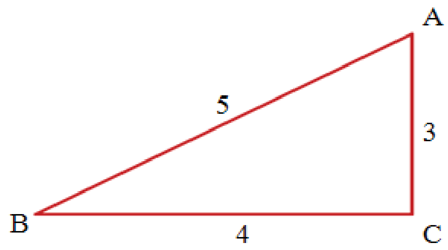
Use $adjascent = 3, opposite = 4\ and\ hypotenuse = 5$

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{4}{5} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{5}{4}$$

$$\cos\theta = \frac{adjascent}{hypotenuse} = \frac{3}{5} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{5}{3}$$

$$\tan\theta = \frac{opposite}{adjascent} = \frac{4}{3} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{3}{4}$$

Example 11



For each angle, calculate the reciprocal trigonometric values.

Angle	$cosecant = \dfrac{1}{sine} = \dfrac{hypotenuse}{opposite}$	$secant = \dfrac{1}{cosine} = \dfrac{hypothenuse}{adjacent}$	$cotangent = \dfrac{1}{tangent}$
A	$\frac{5}{4}$	$\frac{5}{3}$	$\frac{3}{4}$
B	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{4}{3}$
C = 90⁰	$\frac{5}{5}=1$	$\frac{5}{0}$ *which does not exist*	$\frac{0}{5}=0$

Example 12

A positive angle, θ, is in the second quadrant. If $\cos\theta = -\frac{3}{4}$, find the values of the other primary trigonometric values.

Solution

Let h, x *and* y be hypotenuse, adjacent and opposite side respectively.

$\cos\theta = -\frac{3}{4} \Leftrightarrow \frac{x}{h} = -\frac{3}{4}$.

Since $h > 0$, thus $h = 4$ and $x = -3$.

$h^2 = x^2 + y^2 \Rightarrow 16 = 9 + y^2$

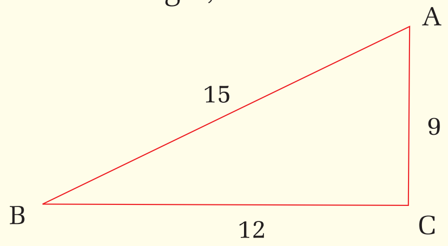
$\Leftrightarrow y^2 = 7 \Rightarrow y = \pm\sqrt{7}$

As θ is in the second quadrant, $y > 0 \Rightarrow y = \sqrt{7}$.

Hence the other primary trigonometric values are $\sin\theta = \frac{y}{h} = \frac{\sqrt{7}}{4}$ and $\tan\theta = \frac{x}{y} = -\frac{\sqrt{7}}{3}$.

Exercise 3

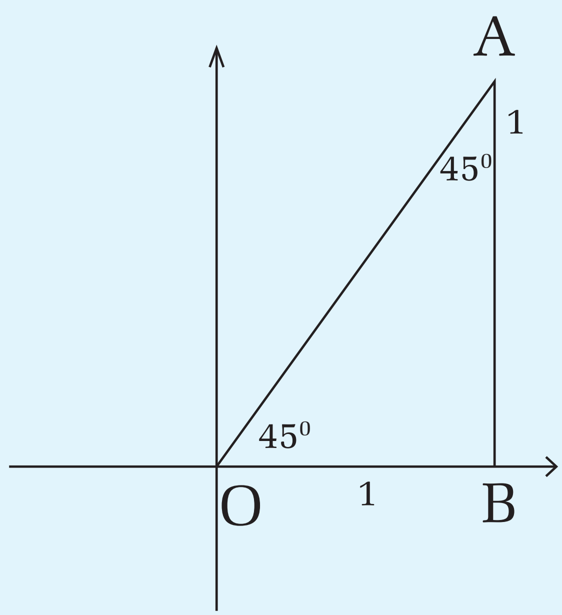
For each angle, calculate the reciprocal trigonometric values



Trigonometric Number of special Angles 30^0, 45^0, 60^0

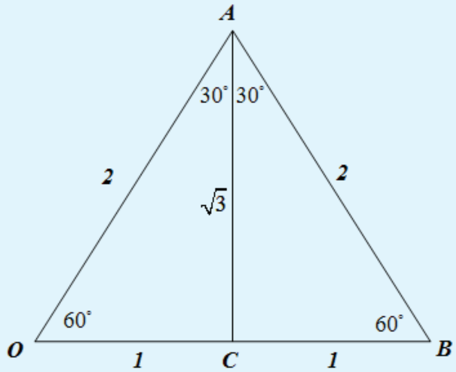
Activity 4

1. **Consider the following diagram**



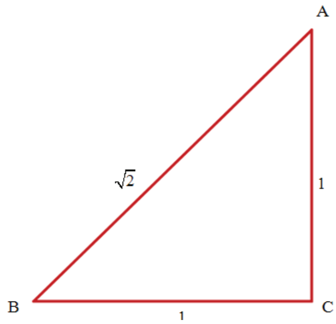
From pythagoras theorem, definition of trigonometric ratios and given diagrams, find $\sin 45^0$, $\cos 45^0$ and $\tan 45^0$

2. Consider the following diagram



- From triangle OAC, find the six trigonometric values of 60^0
- From triangle OAC, find the six trigonometric values of 30^0

As these angles are often used, it is better to keep in your mind their trigonometric ratios in fraction form.

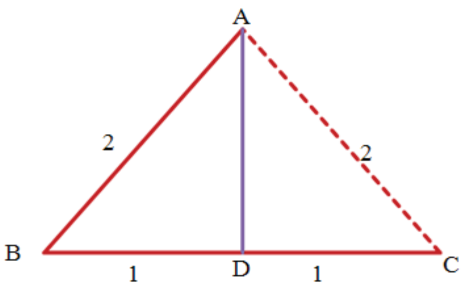


In the above figure, ABC is an isosceles right angled triangle with $\mathrm{BC} = CA = 1$ Hence $AB = \sqrt{2}$ and $\angle A = \angle B = 45^\circ$.

Then

$$\sin 45^0 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \cos 45^0$$

$$\tan 45^0 = 1$$



here on the the figure, ABC is an equilateral triangle with $side\,2$ AD is perpendicular bisector of BC, which implies $BD = 1$ and $AD = \sqrt{3}$. $\angle B = 60^\circ$ .and $\angle BAD = 30^0$.

Then

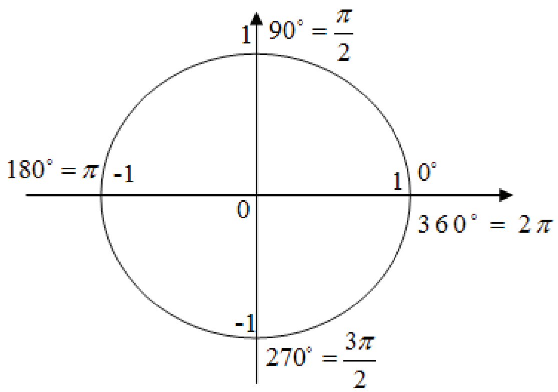
$$\sin 60^0 = \cos 30^0 = \frac{\sqrt{3}}{2}$$

$$\sin 30^0 = \cos 60^0 = \frac{1}{2}$$

$$\tan 60^0 = \sqrt{3},\ \tan 30^0 = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Other special angles

Consider the following unit trigonometric circle.



From the figure, we have;

Angles	0^0	90^0	180^0	270^0	360^0
Sin	0	1	0	-1	0
Cos	1	0	-1	0	1

Exercise 4

Find:

1. $\cot 45^0$
2. $\cot 45^0$
3. $\cot 60^0$
4. $\tan 0^0$
5. $\tan 90^0$
6. $\tan 180^0$
7. $\tan 270^0$
8. $\cot 0^0$
9. $\cot 90^0$
10. $\cot 180^0$
11. $\cot 270^0$

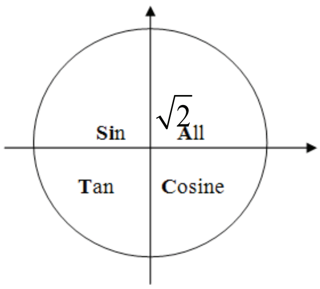
CAST Rule

Activity 5

Copy and complete the table to summarise the signs of the trigonometric values in each quadrant.

	Quadrant			
Value	*I*	*II*	*III*	*IV*
sin	+	?	?	?
cos	+	?	?	?
tan	+	?	?	?

The following diagram shows which primary trigonometric values are positive in each quadrant. This is called the **CAST rule**.



Sine is positive in first and second quadrant but negative in third and fourth quadrant.

Cosine is postive in the first and fouth quadrant but negative in second and third quadrant

Tangent is negative in in the first and third quadrants but negative in second and fourth quadrant.

Exercise 5

State in which of the four quadrants the angles θ must lie, given that

a) $\cos\theta$ is negative and $\sin\theta$ is positive.

b) $\tan\theta$ is negative and $\sin\theta$ is positive.

c) Both $\cot\theta$ and $\csc\theta$ are positive.

d) Both $\cos\theta$ and $\tan\theta$ are positive.

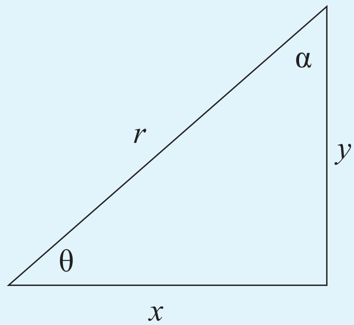
e) $\cot\theta$ is positive and $\csc\theta$ is negative.

f) $\sec\theta$ is negative and $\cot\theta$ is negative.

Trigonometric identities

Activity 6

Here is a right triangle.



In this triangle,

a) Find $\sin\theta$, $\cos\theta$, $\sin\alpha$ and $\cos\alpha$

b) Remembering that on this triangle, Pythagoras' theorem states that

$r^2 = x^2 + y^2$ and dividing each term by r^2 yields

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

So express $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2$ in terms of $\sin\theta$ and $\cos\theta$ and hence, find the values of $\cos^2\alpha + \sin^2\alpha$

Basic Rules

$$\cos^2\theta + \sin^2\theta = 1$$

true for any value of θ

this is the called the **fundamental formula of trigonometry** and is the most frequently used identity in trigonometry.

Dividing this identity by $\cos^2\theta$ *and* $\sin^2\theta$ gives;

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Example 13

Simplify $\dfrac{\csc x}{\sec x}$

Solution

$$\frac{\csc x}{\sec x} = \frac{\frac{1}{\sin x}}{\frac{1}{\cos x}} = \frac{\cos x}{\sin x} = \cot x$$

Example 14

Simplify

$$\left(\frac{1}{\tan x}+\frac{1}{\cot x}\right)\sin x\cos x$$

Solution

$$\left(\frac{1}{\tan x}+\frac{1}{\cot x}\right)\sin x\cos x =$$

$$=\left(\frac{\cos x}{\sin x}+\frac{\sin x}{\cos x}\right)\sin x\cos x$$

$$=\left(\frac{\cos x\cos x+\sin x\sin x}{\sin x\cos x}\right)\sin x\cos x$$

$$=\cos x\cos x+\sin x\sin x$$

$$=\cos^2 x+\sin^2 x$$

$$=1$$

Exercise 6

Simplify

1. $\sec^4 a\left(1-\sin^4 a\right)-2\tan^2 a$
2. $\cot^2 a-\cot^2 b+\dfrac{\sin^2 a-\sin^2 b}{\sin^2 a\sin^2 b}$
3. $\dfrac{\cos^3 a+\cos a\sin^2 a+\sin a}{3\cos^3 a+3\cos a\sin^2 a-\sin a}$

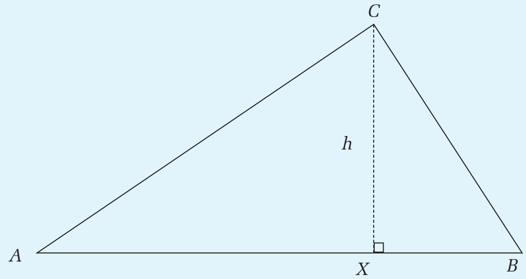
2. Triangle and applications

Solving triangle

Solving a triangle is to find the length of its sides and measures of its angles. There are two methods for solving a triangle: cosine law and sine law.

Cosine law

Activity 7

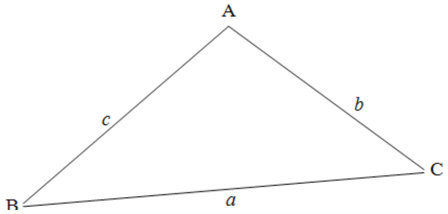


CX is prependicular to side *AB*. Let *AB=c, AC=b, BC=a, CX=h*

1. In triangle *AXC* find cos *A*
2. In triangle *AXC* use pythagoras' theorem to find h^2
3. In triangle *BCX* use pythagoras' theorem to find h^2
4. Combine results obtained in 1, 2 and 3 and give conclusion

Cosine law (also known as cosine formula or cosine rule) relates the lengths of sides to the cosine of one of the angles.

Consider the following triangle:

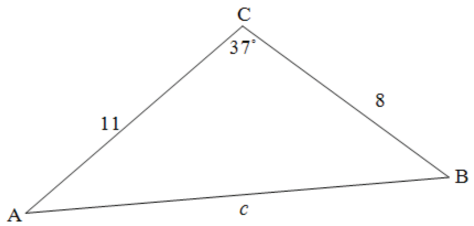


The cosine law says that

$$\begin{cases} a^2 = b^2 + c^2 - 2bc\cos A \\ b^2 = a^2 + c^2 - 2ac\cos B \\ c^2 = a^2 + b^2 - 2ab\cos C \end{cases}$$

Example 15

How long is the side c in the following figure?



Solution

The formula says

$c^2 = a^2 + b^2 - 2ab\cos C$

$a = 8, b = 11\ and\ C = 37^\circ$

$c^2 = 8^2 + (11)^2 - 2\cdot 8\cdot 11\cos 37^\circ$

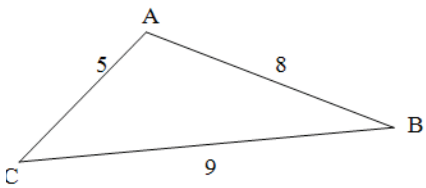
$= 64 + 121 - 176\cdot\cos 37^\circ$

$c = \sqrt{64 + 121 - 176\cdot\cos 37^\circ}$

$= 6.67$ (to 2 decimal places)

Example 16

Find the angle C



Solution

$c^2 = a^2 + b^2 - 2ab\cos C$

$c = 8, a = 9, b = 5$

$8^2 = 9^2 + 5^2 - 2\cdot 9\cdot 5\cos C$

$64 = 106 - 90\cos C$

$\cos C = \frac{42}{90}$

$C = \cos^{-1}\left(\frac{42}{90}\right)$

$= 62.2^\circ$ (to 1 decimal place) *[Since we need the acute angle]*

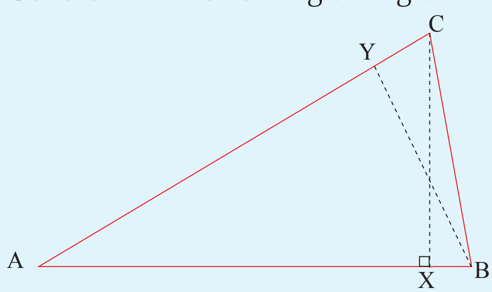
Exercise 7

1. Find the length of the side BC of triangle ABC in which $AB = 7cm, AC = 9cm$ and $\angle BAC = 71^\circ$
2. Find the length of the side AB of triangle ABC in which $BC = 15.3cm, AC = 9.4cm$ and $\angle ACB = 121^\circ$
3. From triangle ABC in which $AC = 9.5cm, BC = 5.5cm$ and $\angle ACB = 145^\circ$, find the value of the angle at A & B then the length of side AB.

Sine law

Activity 8

Consider the following triangle



CX is perpendicular to side AB and BY is perpendicular to side AC. Let AB =c, AC=b, BC=a, CX=h and BY=k

1. In triangle BCX find sin B. In triangle AXC find sin A. Deduce the relationship between side a and side c.
2. In triangle ABY find sin A. In triangle BCY find sine C. Deduce the relationship between side b and side a.
3. Deduce relationship between three sides.

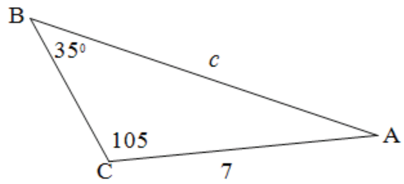
The sine law (or sine formula or sine rule) is an equation relating the lengths of the sides of a triangle to the sine of its angles.

If a,b,c are the lengths of the sides of a triangle and A,B,C are the opposite angles respectively, then the sine law is

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \text{ or } \frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

Example 17

Calculate side c



$$\frac{b}{\sin B}=\frac{c}{\sin C}$$

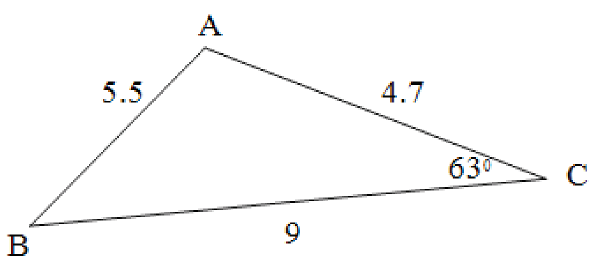
$$\Leftrightarrow \frac{7}{\sin 35^\circ}=\frac{c}{\sin 105^\circ}$$

$$c=\frac{7\sin 105^0}{\sin\ 35^0}$$

$$=11.8 \quad (\text{to } 1 \text{ dp})$$

Example 18

Calculate angle B



$$\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\Leftrightarrow \frac{4.7}{\sin B}=\frac{5.5}{\sin 63^\circ}$$

$$\Leftrightarrow \sin B=\frac{4.7\sin 63^\circ}{5.5}$$

$$B=\sin^{-1}\left(\frac{4.7\sin 63^\circ}{5.5}\right)$$

$$=49.6^\circ \quad (\text{to } 1 \text{ dp})$$

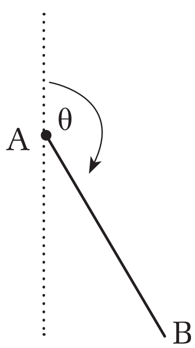
Exercise 8

1. Calculate, in **cm** to 3 significant figures, the length of the sides AB of triangle ABC in which $\angle ACB = 62^\circ$, $\angle ABC = 47^\circ$ and $AC = 7cm$
2. Find, in degrees to 1 decimal place, the size of the angles CAB and ACB in the triangle ABC, where $AC = 4cm$, $BC = 5cm$ and $\angle ABC = 42^\circ$
3. Calculate, in **cm** to 3 significant figures, the length of the sides AC of triangle ABC in which $\angle BAC = 71^\circ$, $\angle ACB = 43^\circ$ and $BC = 6.4cm$

Applications

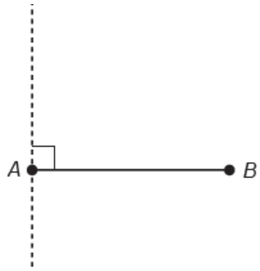
Many real situations involve right triangles. Using angles and trigonometric functions, we can solve problems involving right triangle. We have already seen how to solve a triangle.

1. Bearings and air navigation



We say that point B has a bearing of θ degrees from point A if the line connecting A to B makes an angle of θ with a vertical line drawn through A, the angle being measured clockwise.

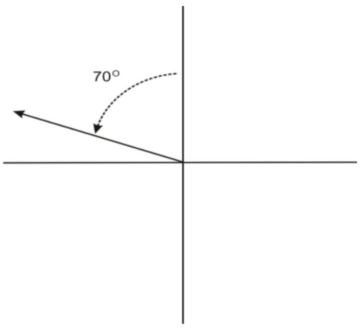
If B is north of A then the bearing is 0^o.



If B is east of A then the bearing is 90°.

Similarly, if B is south of A then the bearing is 180°, and if B is west of A then the bearing is 270°. The bearing can be any number between 0 and 360, because there are 360 degrees in a circle. We can also use right triangles to find distances using angles given as bearings.

In navigation, a bearing is the direction from one object to another. Further, angles in navigation and surveying may also be given in terms of north, east, south, and west. For example, $N70^\circ E$ refers to an angle from the north, towards the east, while $N70^\circ W$ refers to an angle from the north, towards the west.

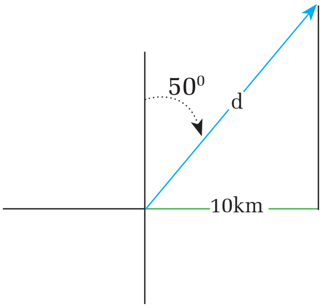


$N70^\circ W$ would result in an angle in the second quadrant.

Example 19

A ship travels on a $N50^\circ E$ course. The ship travels until it is due north of a port which is 10 kilometers due east of the port from which the ship originated. How far did the ship travel?

Solution



The angle between d and 10 kilometres is the complement of 50° which is 40^0.

$$\cos 40^\circ = \frac{adjacent}{hypothenuse} = \frac{10}{d} \Leftrightarrow \cos 40^\circ = \frac{10}{d}$$

$$\Leftrightarrow d\cos 40^\circ = 10$$

$$d = \frac{10}{\cos 40^0}$$

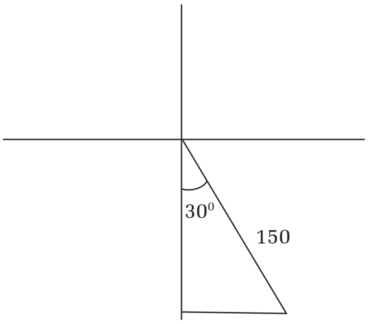
$$\approx 13.05\ km$$

Example 20

An airplane flies on a course of $S30^\circ E$, for 150 km. How far south is the plane from where it originated?

Solution

Using known information, consider the following figure:



$$\cos 30^\circ = \frac{adjacent}{hypothenuse}$$

$$\cos 30^\circ = \frac{y}{150}$$

$$150\cos 30^\circ = y$$

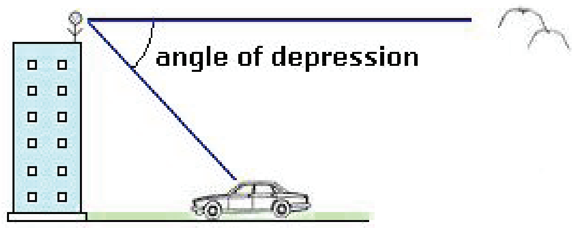
$$y \approx 130\,km$$

Thus, the plane is at *130 km* from where it originated.

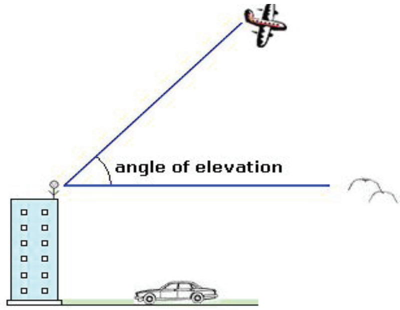
2. Angle of elevation and angle of depression

You can use right triangles to find distances, if you know an angle of elevation or an angle of depression. The figure below shows each of these kinds of angles.

Suppose that an observer is standing at the top of a building and looking straight ahead at the birds (horizontal line). The observer must lower his/her eyes to see the car parked (slanting line). The angle formed between the two lines is called the angle of depression.



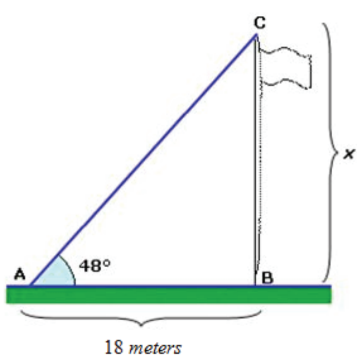
Suppose that an observer is standing at the top of a building and looking straight ahead at the birds (horizontal line). The observer must raise his/her eyes to see the airplane (slanting line). The angle formed between the two lines is called the angle of elevation.



Example 21

The angle of elevation of the top of a pole measures 48° from a point on the ground 18 metres away from its base. Find the height of the flagpole.

Solution



Let x be the height of the flagpole. From the figure

$$\tan 48^\circ = \frac{x}{18}.$$

$$x = 18\tan 48^\circ$$

$$= 19.99$$

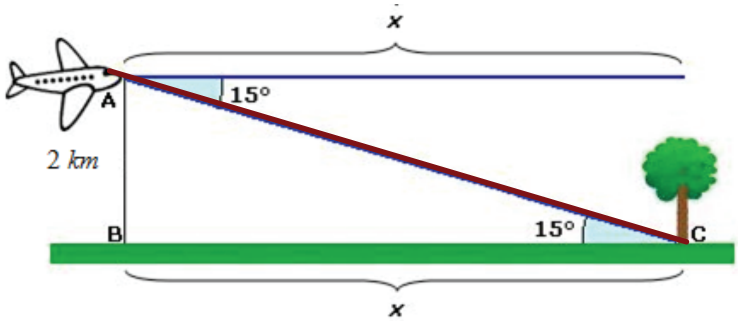
$$\approx 20$$

So, the flagpole is about 20 *metres* high.

Example 22

An airplane is flying at a height of 2 kilometres above the level ground. The angle of depression from the plane to the foot of a tree is 15°. Find the distance that the air plane must fly to be directly above the tree.

Solution



Let x be the distance the airplane must fly to be directly above the tree. The level ground and the horizontal are parallel, so the alternate interior angles are equal in measure.

$$\tan 15^\circ = \frac{2}{x}$$

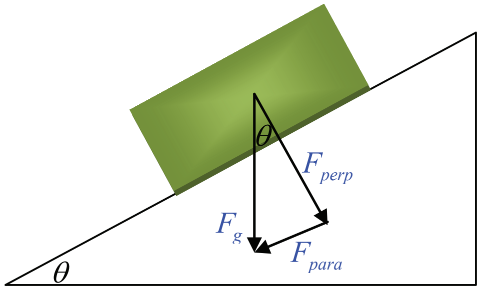
$$x = \frac{2}{\tan 15^\circ}$$

$$\approx 7.46$$

So, the airplane must fly about 7.46 *kilometres* to be directly above the tree.

3. Inclined plane

An **inclined plane**, also known as a **ramp**, is a flat supporting surface tilted at an angle, with one end higher than the other, used as an aid for raising or lowering a load. On the inclined plane the weight of the object causes the object to push into and, the object slides, to rub against the surface of the incline. Also the weight causes the object to be pulled down the slant of the incline. The component that pushes the object into the surface is called the **perpendicular force** $\left(F_{perp}\right)$ and the component that pulls the object down the slanted surface is called the **parallel force** $\left(F_{para}\right)$. The weight vector $\left(F_g\right)$ along its two components F_{perp} and F_{para} form a right triangle. Angle within this right triangle is the same as the angle of the incline, as shown below



From the above figure

$$\sin\theta = \frac{F_{para}}{F_g} \Rightarrow F_{para} = F_g \sin\theta \qquad \cos\theta = \frac{F_{perp}}{F_g} \Rightarrow F_{perp} = F_g \cos\theta$$

In above formula $F_g = mg$ where m is the mass of object and g is the acceleration of gravity $\left(g = 9.8m/s^2\right)$

Then we can write

$$F_{para} = mg\sin\theta$$
$$F_{perp} = mg\cos\theta$$

The unit of force is Newton (N)

Example 23

An object with a mass of 2.5 kg is placed on an inclined plane. The angle of the inclined is 20 degrees. What are the parallel force and the perpendicular force? $\left(g = 9.8m/s^2\right)$

Solution

$$F_{para} = mg\sin\theta$$
$$F_{perp} = mg\cos\theta$$

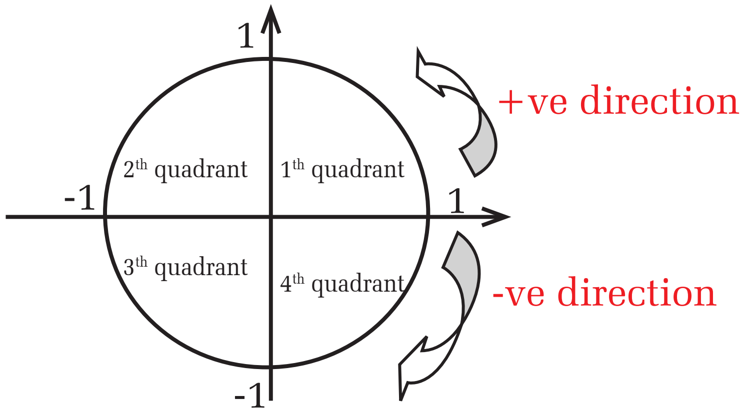
$$F_{para} = 2.5\times 9.8\times \sin 20 \qquad F_{perp} = 2.5\times 9.8\times \cos 20$$
$$= 8.4\,N \qquad\qquad = 23\,N$$

Thus, the parallel force is 8.4 N and the perpendicular force is 23 N.

Unit summary

1. **Trigonometry** is the study of how the sides and angles of a triangle are related to each other. A **rotation angle** is formed by rotating an **initial side** through an angle, about a fixed point called **vertex,** to terminal position called terminal side. Angle is positive if rotated in a counterclockwise direction and negative when rotated clockwise.
2. The amount we rotate the angle is called the measure of the angle and is measured in: degree, grade or radian. $\frac{D}{180}=\frac{R}{\pi}=\frac{G}{200}$, where D stands for degree, R for radians, G for grades and $\pi = 3.14...$
3. In a triangle whose hypotenuse r, the adjacent side x and the opposite side y:
   $$\sin\alpha = \frac{opposite\,side}{hypotenuse}, \cos\alpha = \frac{adjacent\,side}{hypotenuse},\ \tan\alpha = \frac{opposite}{adjascent}$$
   $$\csc\alpha = \frac{hypotenuse}{opposite} = \frac{1}{\sin\alpha}, s\,ec\alpha = \frac{hypotenuse}{adjascent} = \frac{1}{\cos\alpha}$$
   $$and\ \cot\alpha = \frac{adjacent}{opposite} = \frac{1}{\tan\alpha}$$
4. **Unit circle** is a circle of radius one centered at the origin (0,0) in the **Cartesian coordinate system** in the **Euclidian plane.** In the unit circle, the coordinate axes delimit four quadrants that are numbered in an anticlockwise direction.

Each quadrant measures 90 degrees, means that the entire circle measures 360 degrees or radians.



.

5. The table of trigonometric number of remarkable angles

α	0°	30°	45°	60°	90°	180°	270°	360°
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
$\tan\alpha$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	**does not exist**	0	**does not exist**	0
cotan α	**does not exist**	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	**does not exist**	0	**does not exist**

6. Trigonometric identities

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

7. If **a, b, c** are the lengths of the sides of a triangle and **A, B, C** are the opposite angles respectively, then cosine law says that

$$\begin{cases} a^2 = b^2 + c^2 - 2bc\cos\hat{A} \\ b^2 = a^2 + c^2 - 2ac\cos\hat{B} \\ c^2 = a^2 + b^2 - 2ab\cos\hat{C} \end{cases}$$

8. **If** a, b, c **are the lengths of the sides of a triangle and A, B, C are the opposite angles respectively, then the sine law is**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \text{ or } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

9. **Applications**

Many real situations involve right triangle. Using angles and trigonometric functions, we can solve problems involving right triangle like:

- Bearings and air navigation
- Angles of elevation and angle of depression
- Inclined pla

Revision exercise

1. Verify the following identities;

a) $\frac{\sin a}{1-\cos a} = \frac{1+\cos a}{\sin a}$

b) $\sec^2 a + \csc^2 a = (\sec^2 a)\csc^2 a$

c) $\sec^4 a - \tan^4 a = \sec^2 a + \tan^2 a$

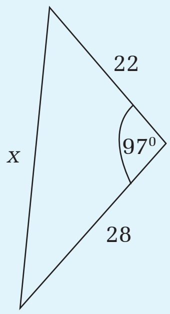
d) $\sqrt{\frac{1-\cos a}{1+\cos a}}$

2. If $\sin\theta = 0.954$ and $\cos\theta = 0.3$, find the value of $\tan\theta$.

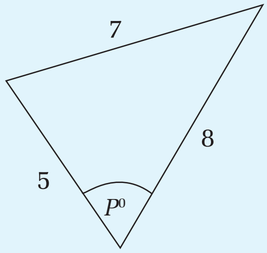
3. If sine A= $\frac{3}{5}$ and A is obtuse, find the values of $\cos A$ and $\tan A$

4. Suppose that the button ***cos*** on your calculator was broken, but the ***sin*** button was working. Explain how you could work out $\cos 14^\circ$

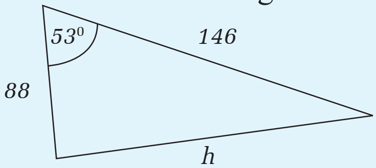
5. Work out the side below;



6. Work out angle $P°$ in the diagram below;



7. Find the missing side in the diagram below



8. A ladder of length 8*m* rests against a wall so that the angle between the ladder and the wall is 31 degrees. How far is the base of the ladder from the wall?
9. A point *P* is 90 *m* away from a vertical flagpole, which is 11 *m* high. What is the angle of elevation to the top of the flagpole from *P*?
10. A ship sails 200 *km* on a bearing of 243.7 degrees
    a) How far south has it travelled?
    b) How far west has it travelled?
11. An aircraft flies 500 *km* on a bearing of 100 degrees and then 600 *km* on a bearing of 160 degrees. Find the distance and bearing of the finishing point from the starting point.
12. A plane is flying at a constant height of 8000 *m*. It flies vertically above me and 30 seconds later the angle of elevation is 74 degrees. Find the speed of the plane in metres/second.
13. Convert $81^0 13' 08''$ to decimal degree
14. Convert 117.6572^0 to d^0 m' s'' system
15. Convert 2.937^0 to d^0 m' s'' system
16. Convert $75^0 19' 35''$ to the nearest tenth degree

Unit 2

Set IR of real numbers

My goals

By the end of this unit, i will explain:

- Absolute value and its properties.
- Powers and radicals.
- Decimal logarithms and properties.

Introduction

Real numbers are all Rational and Irrational numbers. They can also be positive, negative or zero. A simple way to think about the Real Numbers is: any point anywhere on the number line. Powers (convenient way of writing multiplications that have many repeated terms), radicals (an expression that has a square root, cube root, …), a logarithm (the power to which a number must be raised in order to get some other number) of a number …. are all real numbers.

Suppose that your land has the form of a square and you need to find its area. What you need to do is to multiply one side of the land by itself which gives *(side)*^{2} . Also, suppose that you need to find the amount of water are in cuboid container. What you need to do is to multiply one edge of the container by itself three times which gives *(edge)*^{3}. *(side)*^{2} and *(edge)*^{3} are powers. In economics, to calculate the inflation of a home that increases in value from P_1 to P_2 over a period of n years, the annual rate of inflation is $i = \sqrt[n]{\frac{p_1}{p_2}} - 1$. The expression $\sqrt[n]{\frac{p_1}{p_2}}$ is a radical form.

If you have bacteria that divide every 40 minutes and are currently taking up 0.1% of the petri dish, you can use logarithms to estimates how long it will take them to fill up the entire dish. The same goes for 2000 francs in an account, logarithms will tell you when you will have 4000 francs. In finance and business logarithms can be useful for calculating compound interest.

Required outcomes

After completing this unit, the learners should be able to:

- Define absolute value of a real number and solve simple equations involving absolute value.
- Define powers and their properties.
- Define radicals and their properties.
- Define decimal logarithm of a real number and solve simple logarithmic equations.

1. Absolute value and its properties

Activity 1

Draw a number line and state the number of units that are between;

1. 0 and -8
2. 0 and 8
3. 0 and $\frac{1}{2}$
4. 4 and 17

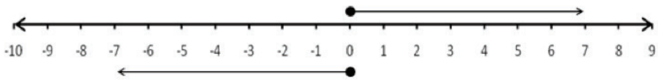
Absolute value of a number is the distance of that number from the original (zero point) on a number line. The symbol

$|\ |$ is used to denote the absolute value.

Example 1

7 is at 7 units from zero, thus the absolute value of 7 is 7 or

$|7| = 7$. Also -7 is at 7 units from zero, thus the absolute value of -7 is 7 or $|-7|=7$. So $|-7|=|7|=7$ since -7 and 7 are on equal distance from zero on a number line.



Note:

- The absolute value of zero is zero.
- The absolute value of a non-zero real number is a positive real number.
- Given that $|x|=k$ where k is a positive real number or zero, then $x=-k$ *or* $x=k$
- $|x|=\begin{cases} x, & if\ x \geq 0 \\ -x, & if\ x<0 \end{cases}$

Example 2

Find x in the following

a) $|x|=5$ b) $|x|+5=1$ c) $|x-4|=10$

Solution

a) $|x|=5,\ x=-5$ *or* $x=5$

b) $|x|+5=1$

$\Leftrightarrow |x|=1-5 \Rightarrow |x|=-4$

There is no value of x since the absolute value of x must be a positive real number.

c) $|x-4|=10$

$x-4=-10$ *or* $x-4=10$

$x=-10+4$ *or* $x=10+4$

$x=-6$ *or* $x=14$

Example 3

Simplify:

a) $-|40-12|$ b) $|4(-3)-(2)(5)|$ c) $|-4(-2)|$

Solution

a) $-|40-12| = -|28| = -28$

b) $|4(-3)-(2)(5)| = |-12-10| = |-22| = 22$

c) $|-4(-2)| = |8| = 8$

Exercise 1

Find the value(s) of x

1) $|x| = 6$ 2) $|x+1| = 0$

3) $|x-3|-4 = 2$ 4) $|2x+1| = 4$ 5) $|x-3|+3 = 5$

Activity 2

Evaluate the following operations and compare your results in eache case

1. $|3|$ and $|-3|$ 2. $|3\times 5|$ and $|3|\times|5|$ 3. $|(-8)+5|$ and $|-8|+|5|$

Properties of the Absolute Value

Opposite numbers have equal **absolute value**.

$$|a| = |-a|$$

Example 4

$$|5| = |-5| = 5$$

The absolute value **of a product** is equal to the **product of the absolute values** of the factors.

$$|ab| = |a||b|$$

Example 5

$$|4(-6)| = |4||-6|$$

$|4(-6)| = |-24| = 24$ $\qquad$ $|4||-6| = 4 \times 6 = 24$

The absolute value **of a sum is less than or equal to the sum of the absolute values of the addends**.

$$|a+b| \leq |a| + |b|$$

Example 6

$$|-3+2| \leq |-3| + |2|$$
$$|-1| \leq 3+2$$
$$1 \leq 5$$

Exercise 2

Simplify:

1. $|-5|$
2. $|-4||-5|$
3. $|-7| + |4|$
4. $-|4 \times 6|$
5. $-|-6+8|$

2. Powers and radicals

Powers in IR

Activity 3

Peter suggested that his allowance be changed. He wanted \$2 the first week, with his allowance to be doubled each week. His parent investigated the suggestion using this table

Week	Dollas
One	2=...
Two	$2\times 2=\dots$
Three	$2\times 2\times 2=\dots$
Four	$2\times 2\times 2\times 2=\dots$
Five	$2\times 2\times 2\times 2\times 2=\dots$

1. Complete the table to find how many dollars Peter would be paid each of the first five weeks.
2. How much would Peter be paid the seventh week? The tenth week?
3. Do you think his parent will agree with his suggestion? Explain.

We call n^{th} power of a real number a that we note a^n , the product of n factors of a.

That is

$$a^n = \underbrace{a\cdot a\cdot a\ldots a}_{n\ factors}\quad \begin{cases} n \text{ is an exponent} \\ a \text{ is the base}\end{cases}$$

Example 7

$$2^4 = \underbrace{2\cdot 2\cdot 2\cdot 2}_{4\ factors} = 16 \qquad 3^3 = \underbrace{3\cdot 3\cdot 3}_{3\ factors} = 27$$

Notice

- $a^1 = a$
- $a^0 = 1, a \neq 0$
- If $a = 0, a^0$ is not defined

Properties of powers

Let $a,b\in\mathbb{R}$ and $m,n\in\mathbb{R}$

a) $a^m\cdot a^n = a^{m+n}$

In fact, $a^m\cdot a^n = \underbrace{a\cdot a\cdot a\cdots a}_{m\ factors}\times\underbrace{a\cdot a\cdot a\cdots a}_{n\ factors} = \underbrace{a\cdot a\cdot a\cdots a}_{m+n\ factors} = a^{m+n}$

b) $\left(a^m\right)^n = a^{mn}$

In fact, $\left(a^m\right)^n = \underbrace{\underbrace{a\cdot a\cdot a\cdots a}_{m\ factors}\times\underbrace{a\cdot a\cdot a\cdots a}_{m\ factors}\cdots\underbrace{a\cdot a\cdot a\cdots a}_{m\ factors}}_{n\ factors} = a^{mn}$

c) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

In fact, $\left(\frac{a}{b}\right)^m = \underbrace{\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}\cdots\frac{a}{b}}_{m\ factors} = \frac{\overbrace{a\cdot a\cdot a\cdots a}^{m\ factors}}{\underbrace{b\cdot b\cdot b\cdots b}_{m\ factors}} = \frac{a^m}{b^m}$

d) $\frac{1}{b^m} = b^{-m}$

In fact, $\frac{1}{b^m} = \frac{1}{b^m}^m = \left(\frac{1}{b}\right)^m = \left(b^{-1}\right)^m = b^{-m}$

e) $\frac{a^m}{a^n} = a^{m-n}$

In fact, $\frac{a^m}{a^n} = a^m\frac{1}{a^n} = a^m a^{-n} = a^{m-n}$

f) $(ab)^m = a^m b^m$

In fact, $(ab)^m = \underbrace{ab\cdot ab\cdots ab}_{m\ factors} = \underbrace{a\cdot a\cdots a}_{m\ factors}\times\underbrace{b\cdot b\cdots b}_{m\ factors} = a^m b^m$

These properties help us to simplify some powers.

There is no general way to simplify the sum of powers, even when the powers have the same base. For instance, $2^5 + 2^3 = 32 + 8 = 40$ and 40 is not an integer power of 2. But some products or ratios of powers can be simplified using repeated multiplication model of a^n.

Example 8

a) $2^4 \cdot 2^3 \cdot 4 = 2^4 \cdot 2^3 \cdot 2^2 = 2^9 = 512$

b) $a^4 \cdot b^3 \cdot a^5 \cdot b^8 = a^4 \cdot a^5 \cdot b^3 \cdot b^8 = a^9 \cdot b^{11}$

$a^9 \cdot b^{11}$ cannot be simplified further because the bases are different.

c) $\dfrac{y^9}{y^2} = y^{9-2} = y^7$

Exercise 3

Simplify

1. $x^3 x^2$
2. $\left(xy^3\right)^2 + 4x^2 y^6$
3. $\dfrac{6xy^2}{3xy}$
4. $\dfrac{ab}{a^3} - \dfrac{a^3 b^2}{a^5 b}$
5. $\dfrac{yx}{4xy}$

Radicals in real numbers

Activity 4

Evaluate the following powers

1. $(81)^{\frac{1}{2}}$
2. $(216)^{\frac{1}{3}}$
3. $(-27)^{\frac{1}{3}}$
4. $(16)^{\frac{1}{4}}$

The n^{th} root of a real number is $\dfrac{1}{n}$ power of that real number. It is denoted by $\sqrt[n]{b}$, $b \in \mathbb{R}, n \in \mathbb{N} \setminus \{1\}$.

$\forall a, b \in \mathbb{R}, \sqrt[n]{b} = a \Leftrightarrow b^{\frac{1}{n}} = a \Leftrightarrow b = a^n$

$$\begin{cases} n \text{ is called the index} \\ b \text{ is called the base or radicand} \\ \sqrt[n]{\ } \text{ is called the radical sign} \end{cases}$$

Example 9

a) $\sqrt[3]{27} = (27)^{\frac{1}{3}} = \left(3^3\right)^{\frac{1}{3}} = 3^{3\times\frac{1}{3}} = 3$

b) $\sqrt[4]{16} = (16)^{\frac{1}{4}} = \left(2^4\right)^{\frac{1}{4}} = 2$

If $n = 2$, we say square root and $\sqrt[2]{b}$ is written as $\sqrt{b}$. Here b must be a positive real number or zero.

If $n = 3$, we say cube root denoted by $\sqrt[3]{b}$. Here b can be any real number.

If $n = 4$, we say 4th root denoted by $\sqrt[4]{b}$. Here b must be a positive real number or zero.

Generally, for any natural number $n \geq 2$, n^{th} root of b is denoted as $\sqrt[n]{b}$. Here if n is even, b must be a positive real number or zero and if n is odd b can be any real number.

Example 10

$\sqrt{-9}$ is not defined in $\mathbb{R}$ since the index in radical is even.

But $\sqrt[3]{-27} = (-27)^{\frac{1}{3}} = \left[(-3)^3\right]^{\frac{1}{3}} = -3$

Properties of radicals

$\forall n \in \mathbb{N} \setminus \{1\}, m \in \mathbb{R}$

a) $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

In fact, $\sqrt[n]{a^m} = \left(a^m\right)^{\frac{1}{n}} = a^{m\times\frac{1}{n}} = a^{\frac{m}{n}}$

b) $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

In fact, $\sqrt[n]{ab} = (ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}} = \sqrt[n]{a}\sqrt[n]{b}$

c) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

In fact, $\sqrt[n]{\frac{a}{b}} = \left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

d) $\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a} = a^{\frac{1}{nm}}$

In fact, $\sqrt[n]{\sqrt[m]{a}} = \left(a^{\frac{1}{m}}\right)^{\frac{1}{n}} = a^{\frac{1}{m}\times\frac{1}{m}} = a^{\frac{1}{mn}} = \sqrt[mn]{a}$

Example 11

Simplify:

a) $\sqrt{46656}$ b) $\sqrt[3]{\sqrt{64}}$ c) $\sqrt[3]{ab}\times\sqrt[3]{a^2b^2}$ d) $\sqrt{\frac{36}{81}}$

Solution

a) $\sqrt{46656} = \sqrt{6^6} = 6^3 = 216$

b) $\sqrt{\sqrt{64}} = \sqrt{64} = \sqrt{2} = 2$

c) $\sqrt[3]{ab}\times\sqrt[3]{a^2b^2} = \sqrt[3]{a^3b^3} = \sqrt[3]{(ab)} = ab$

d) $\sqrt{\frac{36}{81}} = \frac{\sqrt{36}}{\sqrt{81}} = \frac{6}{9} = \frac{2}{3}$

Exercise 4

Simplify:

1. $\sqrt{a}\times\sqrt{ab^3}\times\sqrt{bc^2}$
2. $\sqrt[3]{abc}\times\sqrt[3]{a^2b^2c^2}$
3. $\sqrt[3]{\frac{8}{27}}$
4. $\sqrt{\sqrt[4]{x^8}}$
5. $\sqrt{\frac{x^3y^4}{4x}}$

Operations on radicals

When adding or subtracting the radicals, we may need to simplify if we have similar radicals. Similar radicals are the radicals with the same indices and same bases.

Activity 5

Simplify the following and keep the answer in radical sign

1. $\sqrt{18}+\sqrt{2}$
2. $\sqrt{12}-3\sqrt{3}$

3. $\sqrt{2}\times\sqrt{3}$ 4. $\dfrac{\sqrt{6}}{\sqrt{2}}$

Addition and subtraction

When adding or subtracting the radicals we may need to simplify if we have similar radicals. Similar radicals are the radicals with the same indices and same bases.

Example 12

$$\sqrt{2}+\sqrt{8}=\sqrt{2}+\sqrt{2\times4}=\sqrt{2}+\sqrt{2}\times\sqrt{4}=\sqrt{2}+2\sqrt{2}=3\sqrt{2}$$

$$\sqrt{3}-\sqrt{27}=\sqrt{3}-\sqrt{3\times9}=\sqrt{3}-\sqrt{3}\times\sqrt{9}=\sqrt{3}-3\sqrt{5}=-2\sqrt{3}$$

Exercise 5

Simplify

1. $\sqrt{20}+\sqrt{5}$
2. $4\sqrt{3}-\sqrt{12}$
3. $5\sqrt{7}-\sqrt{28}$
4. $\sqrt{18}\times\sqrt{8}$
5. $\sqrt{45}+\sqrt{80}+\sqrt{180}$
6. $\sqrt{108}-\sqrt{48}$

Rationalizing

Activity 6

Make the denominator of each of the following rational;

1. $\dfrac{1}{\sqrt{2}}$
2. $\dfrac{2-\sqrt{3}}{2\sqrt{5}}$
3. $\dfrac{2}{1-\sqrt{6}}$
4. $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{3}+\sqrt{5}}$

Rationalizing is to convert a fraction with an irrational denominator to a fraction with rational denominator. To do this, if the denominator involves radicals, we multiply the numerator and denominator by the conjugate of the denominator.

The conjugate of $a\pm\sqrt{b}$ is $a\mp\sqrt{b}$.

The conjugate of $\sqrt{a}$ is $\sqrt{a}$

The conjugate of $a\sqrt{b}$ is $\sqrt{b}$

Remember that $(a+b)(a-b)=a^2-b^2$

Example 13

a) $\frac{1}{1+\sqrt{2}}=\frac{1-\sqrt{2}}{(1+\sqrt{2})(1-\sqrt{2})}=\frac{1-\sqrt{2}}{1-2}=\frac{1-\sqrt{2}}{-1}=\sqrt{2}+1$

b) $\frac{\sqrt{2}}{\sqrt{5}-\sqrt{3}}=\frac{\sqrt{2}(\sqrt{5}+\sqrt{3})}{(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})}=\frac{\sqrt{10}+\sqrt{6}}{5-3}=\frac{\sqrt{10}+\sqrt{6}}{2}$

c) $\frac{\sqrt{3}+\sqrt{7}}{4\sqrt{2}}=\frac{(\sqrt{3}+\sqrt{7})\sqrt{2}}{(4\sqrt{2})\sqrt{2}}=\frac{\sqrt{6}+\sqrt{14}}{8}$

Exercise 6

Rationalize the denominator;

1. $\frac{5}{\sqrt{7}}$
2. $\frac{3-2\sqrt{2}}{1-\sqrt{2}}$
3. $\frac{2\sqrt{6}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$
4. $\frac{2\sqrt{2}}{4+3\sqrt{3}}$
5. $\frac{a-\sqrt{b}}{\sqrt{d}}$
6. $\frac{3\sqrt{3}+2\sqrt{2}}{1+2\sqrt{2}}$

3. Decimal logarithms and properties

Activity 7

What is the real number for which 10 must be raised to obtain

1. 1
2. 10
3. 100
4. 1000
5. 10000
6. 100000

The **decimal logarithm** of a positive real number x is defined to be a real number y for which 10 must be raised to obtain x.

We write $\forall x > 0, \quad y = \log x$

$\log x$ is the same as $\log_{10} x$ and is defined for all positive real numbers only. 10 is the base of this logarithm. In general notation, we do not write this base for decimal logarithm.

In the notation $y = \log x$, x is said to be the **antilogarithm of** y.

Example 14

log (100) = ?

We are required to find the power to which 10 must be raised to obtain 100

So log(100) = 2

$y = \log x$ means $10^y = x$

Be Careful! $\log 2x + 1 \neq \log(2x+1)$

$\log 2x + 1 = (\log 2x) + 1$

Since logs are defined using exponentials, any "log x" has an equivalent "exponent" form, and vice-versa.

Example 15

$\log_5 13 = x \Leftrightarrow 5^x = 13$

Example 16

$y^{14} = x \Leftrightarrow \log_y x = 14$

Properties

$\forall a, b \in]0, +\infty[$

a) $\log ab = \log a + \log b$

b) $\log \frac{1}{b} = -\log b$

c) $\log \frac{a}{b} = \log a - \log b$

d) $\log a^n = n \log a$

Example 17

Calculate in function of $\log a, \log b\ and \log c$

a) $\log a^2b^2$ b) $\log \frac{ab}{c}$

Solution

a) $\log a^2b^2 = \log(ab)^2$

$= 2\log ab$

$= 2(\log a + \log b)$

b) $\log \frac{ab}{c} = \log ab - \log c$

$= \log a + \log b - \log c$

Example 18

Given that $\log 2 = 0.30$, $\log 3 = 0.48$ and $\log 5 = 0.69$. Calculate

a) $\log 6$ b) $\log 0.9$

Solution

a) $\log 6 = \log(2 \times 3)$

$= \log 2 + \log 3$

$= 0.30 + 0.48$

$= 0.78$

b) $\log 0.9 = \log \frac{9}{10}$

$= \log 9 - \log 10$

$= \log 3^2 - \log(2 \times 5)$

$= 2\log 3 - \log 2 - \log 5$

$= 2(0.48) - 0.30 - 0.7$

$= -0.04$

Co-logarithm

Co-logarithm, sometimes shortened to **colog**, of a number is the logarithm of the reciprocal of that number, equal to the negative of the logarithm of the number itself,

$$\text{colog}\, x = \log\left(\frac{1}{x}\right) = -\log x$$

Example 19

$\text{colog } 200 = -\log\ 200 = -2.3010$

Exercise 7

1. Without using calculator, compare the numbers a and b.
   a) $a = 3\log 2$ and $b = \log 7$
   b) $a = 2\log 2$ and $b = \log 16 - \log 3$
2. Given that $\log 2 = 0.30$, $\log 3 = 0.48$ and $\log 5 = 0.69$. Calculate
   a) $\log 150$ b) $\log 0.2 + \log 10$
3. Find co-logarithm of:
   a) 100   b) 42   c) 15

Unit summary

1. Absolute value of a number is the number of units it is from 0 on a number line. The symbol $|\ |$ is used to denote the absolute value.
2. We call n^{th} power of a real number a that we note a^n, the product of n factors of a. that is

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n\ factors} \quad \begin{cases} n\ is\ an\ exponent \\ a\ is\ the\ base \end{cases}$$

3. The n^{th} root of a real number is $\frac{1}{n}$ power of that real number. It is noted by $\sqrt[n]{b}$, $b \in \mathbb{R}, n \in \mathbb{N} \setminus \{1\}$.

$$\forall a, b \in \mathbb{R}, \sqrt[n]{b} = a \Leftrightarrow b^{\frac{1}{n}} = a \Leftrightarrow b = a^n$$

$$\begin{cases} n\ is\ called\ the\ index \\ b\ is\ called\ the\ base\ or\ radicand \\ \sqrt[n]{\ }\ \ is\ called\ the\ radical\ sign \end{cases}$$

4. Rationalizing is to convert a fraction with an irrational denominator to a fraction with rational denominator. To do this, if the denominator involves radicals we multiply

the numerator and denominator by the conjugate of the denominator.

5. The **decimal logarithm** of a positive real number x is defined to be a real number y for which 10 must be raised to obtain x. We write $\forall x > 0, \quad y = \log x$

Revision exercise

1. Simplify:

   a) $\frac{xy^2z}{xy}$ b) $(ab^2)^3 + a^3b^6$ c) $\sqrt{2} - \sqrt{8} + \sqrt{18}$

2. Rationalize the denominator

   a) $\frac{3\sqrt{5}+\sqrt{2}}{2\sqrt{7}}$ b) $\frac{\sqrt{5}+\sqrt{2}}{2-\sqrt{3}}$ c) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}+\sqrt{5}}$

3. Given that $\log 2 = 0.30$, $\log 3 = 0.48$ and $\log 5 = 0.69$. Calculate

   a) $\log 12$ b) $\log 0.45$ c) $\log \frac{18}{5}$

4. Match the fact with the power

   a) Wheels on a unicycle — i. 2^5

   b) Planets in the solar system — ii. 3^2

   c) Freezing point of water in — iii. 1^{17}

5. Contractors are tilling the bathroom floor in new house. The floor measures 288 cm by 192 cm. They are using square tiles with sides measuring 24 cm. How many tiles will they need?

6. Esther's little brother is playing with a set of collared blocks. Each block has edges measuring 4 cm. What is the volume of one of the blocks?

Unit 3

Linear, quadratic equations and inequalities

My goals

By the end of this unit, I will explain:

- Equations and inequalities in one unknown.
- Simultaneous equations in two unknowns.
- Applications.

Introduction

An equation is a statement that the values of two mathematical expressions are equal while an inequality is a statement that the values of two mathematical expressions that are not equal.

Lots of students are interested in aviation. Would you like to be a private pilot and fly your own small plane? You can't pass the test to get a pilot's license unless you can work out equations and calculate how to load the plane correctly with people and baggage. You should already understand what it means to make a profit - you sell an item for more than it cost to make it. So in very simple terms, companies use this equation all the time: **Profit = selling price - cost**. There are many financial decisions we make everyday based on how much we earn, less all our expenses, then we say we can spend up to x on a new purchase or you put x in a saving account, or something else. These are all inequalities.

A quadratic equation is an equation of degree 2, meaning that the highest exponent is 2. In daily life, say that you want to throw a ball into the air and have your friend catch it, but you want to give her/him the precise time it will take the ball to arrive.

To do this, you would use the velocity equation, which calculates the height of the ball based on a parabolic (quadratic) equation. Quadratics are equations and inequalities are useful in calculating areas, figuring out profits, finding speeds, athletics,...

Required outcomes

After completing this unit, the learners should be able to:

- Solve equation of the first degree and second degree.
- Solve inequality of the first degree and second degree.
- Solve a system of linear equations.
- Use equations and inequalities to solve word problems.
- Apply equations and inequalities in real life problems.

1. Equations and inequalities in one unknown

Equations

Activity 1

Find the value of x such that the following statements are true;

1. $x+1=5$
2. $2x-4=0$
3. $2x+1=-5$
4. $x+34=0$
5. $x-1=2$
6. $x-4=10$

An equation is a statement that the values of two mathematical expressions are equal. Consider the statement $x-3=0$. This statement is true when $x=3$. So $x=3$ is called the solution of the statement $x-3=0$. The number 3 is called the root of the equation. Thus, to find a solution to the given equation is to find the value that satisfies that equation.

To do this, rearrange the given equation such that variables will be in the same side and constants in the other side and then find the value of the variable.

Example 1

Solve in set of real numbers:

a) $x+6=14$ b) $4x+5=20+x$ c) $x=14-x$

Solution

a) $x+6=14$

$\Leftrightarrow x=14-6$

$\Rightarrow x=8$

$S=\{8\}$

b) $4x+5=20+x$

$\Leftrightarrow 4x-x=20-5$

$\Leftrightarrow 3x=15$

$\Leftrightarrow x=\frac{15}{3}$

$\Rightarrow x=5$

$S=\{5\}$

c) $x=14-x$

$\Leftrightarrow 2x=14$

$\Rightarrow x=7$

$S=\{7\}$

Exercise 1

Solve in set of real numbers:

1. $x+5=9$
2. $6x+5=5$
3. $x-2=3$
4. $25=2x-5$
5. $-5=x-1$
6. $3x-4=2x+1$
7. $x+5=9x+1$
8. $-6x-5=9$
9. $x+100=99$
10. $6x-51=9$

Equations products / quotients

Activity 2

State the method you can use to solve the following equations

1. $(x+1)(x-1)=0$ 2. $(2x-3)x=0$ 3. $\frac{2x-3}{x}=\frac{1}{2}$

When we are given the equation $A \cdot B = 0$ then $A = 0$ or $B = 0$.

Also $\frac{A}{B} = \frac{C}{D} \Leftrightarrow A \cdot D = B \cdot C$

Example 2

Solve in set of real numbers

a) $(3x+6)(x-5)=0$ b) $(-x+2)(2x+3)=0$ c) $\frac{2x+5}{x-6}=4, x \neq 6$

Solution

a) $(3x+6)(x-5)=0$

$3x+6=0$ or $x-5=0$

$x=-2$ or $x=5$

$S=\{-2,5\}$

b) $(-x+2)(2x+3)=0$

$-x+2=0$ or $2x+3=0$

$x=2$ or $x=-\frac{3}{2}$

$S=\left\{-\frac{3}{2},2\right\}$

c) $\frac{2x+5}{x-6}=4, x \neq 6$

$\Leftrightarrow 2x+5=4(x-6)$

$\Leftrightarrow 2x+5=4x-24$

$\Leftrightarrow -2x=-29$

$\Rightarrow x=\frac{29}{2}$

$S=\left\{\frac{29}{2}\right\}$

Exercise 2

Solve in set of real numbers:

a) $(3x+6)(x-5)=0$ b) $\frac{3}{x-6}=\frac{4}{2x-5}$

c) $(x+8)(2x+1)=0$ d) $\frac{-x+5}{x-3}=\frac{4}{7}$

Inequalities

Activity 3

Find the value(s) of x such that the following statements are true

1. $x < 5$ 2. $x > 0$ 3. $-4 < x < 12$ 4. $x \le 100$

Suppose that we have the inequality $x+3<10$. In this case, we have an inequality with one unknown. Here, the real value of x satisfies that this inequality is not unique. For example, 1 is a solution but 3 is also a solution. In general, all real numbers less than 7 are solutions. In this case, we will have many solutions combined in an interval.

Now, the solution set of $x+3<10$ is an open interval containing all real numbers less than 7 whereby 7 is excluded. How?

We solve this inequality as follows;

$x+3<10$

$\Leftrightarrow x<10-3$

$\Leftrightarrow x<7$

And then $S =]-\infty, 7[$

Recall that

- When the same real number is added or subtracted from each side of the inequality, the direction of the inequality is not **changed**.
- The direction of the inequality is not **changed** if both sides are multiplied or divided by the same **positive real number** and is **reversed** if both sides are multiplied or divided by the **same negative real** number.

Example 3

Solve in set of real numbers:

a) $-2x+5\le 0$ b) $x-4>0$

c) $2(x+5)>2x-8$ d) $2x+5\le 2x+4$

Solution

a) $-2x+5\le 0$

$\Leftrightarrow -2x\le -5$

$\Leftrightarrow x\ge \frac{5}{2}$ $\qquad S=\left[\frac{5}{2},+\infty\right[$

b) $x-4>0$

$\Leftrightarrow x>4$ $\qquad S=\left]4,+\infty\right[$

c) $2(x+5)>2x-8$

$\Leftrightarrow 2x+10>2x-8$

$\Leftrightarrow 0x>-18$

Since any real number times zero is zero and zero is greater than -18, then the solution set is the set of real numbers.

$S=\mathbb{R}=\left]-\infty,+\infty\right[$

d) $2x+5\le 2x+4 \qquad 0x\le -1$

Since any real number times zero is zero and zero is not less or equal to, -1 then the solution set is the empty set.

$S=\varnothing$

Exercise 3

Solve the following inequalities

1) $x+6<15$
2) $2x-4<16$
3) $5x\le 25$
4) $3x-5>21$
5) $2x+8\ge 18$
6) $6+x<10$
7) $5x\le 5x+2$
8) $3x-5>2+3x$
9) $2x+1\ge 12+3x$
10) $6-x<9$

Inequalities products / quotients

Activity 4

State the method you can use to solve the following inequalities

1. $(x+1)(x-1)<0$
2. $\frac{2x-3}{x}<0$

Suppose that we need to solve the inequality of the form $(ax+b)(cx+d)<0$. For this inequality, we need the set of all real numbers that make the left hand side to be negative. Suppose also that we need to solve the inequality of the form $(ax+b)(cx+d)>0$. For this inequality, we need the set of all real numbers that make the left hand side to be positive.

We follow the following steps:

a) First we solve for $(ax+b)(cx+d)=0$

b) We construct the table called sign table, find the sign of each factor and then the sign of the product or quotient if we are given a quotient.

For the quotient, the value that makes the denominator to be zero is always excluded in the solution. For that value, we use the symbol $||$ in the row of quotient sign.

c) Write the interval considering the given inequality sign.

Example 4

Solve in set of real numbers;

a) $(3x+7)(x-2)<0$ b) $\frac{x+4}{2x-1}\geq 0$

Solution

a) $(3x+7)(x-2)<0$

Start by solving $(3x+7)(x-2)=0$

$3x+7=0$ $\Leftrightarrow x=-\frac{7}{3}$ or $x-2=0$ $\Leftrightarrow x=2$

Then next is to find the sign table.

x	$-\infty$		$-\frac{7}{3}$		2		$+\infty$
$3x+7$		-	0	+		+	

$x-2$	-		-	0	+
$(3x+7)(x-2)$	+	0	-	0	+

Since the inequality is $(3x+7)(x-2)<0$, we will take the interval where the product is negative. Thus, $S=\left]-\frac{7}{3},2\right[$

b) $\frac{x+4}{2x-1}\geq 0$ $\quad x+4=0 \Rightarrow x=-4 \quad 2x-1=0 \Rightarrow x=\frac{1}{2}$

x	$-\infty$		-4		$\frac{1}{2}$		$+\infty$
$x+4$		-	0	+		+	
$2x-1$		-		-	0	+	
$\frac{x+4}{2x-1}$		+	0	-	$\|$	+	

$$S=\left]-\infty,-4\right]\cup\left]\frac{1}{2},+\infty\right[$$

Exercise 4

Solve the following inequalities:

1. $(x-3)(x+3)>0$
2. $(4x-3)(x-1)\leq 0$
3. $(x-2)(-x-5)(-x-1)<0$
4. $\frac{3x+4}{-x-1}>0$
5. $\frac{2x-6}{x+2}\geq 0$

Inequalities involving absolute value

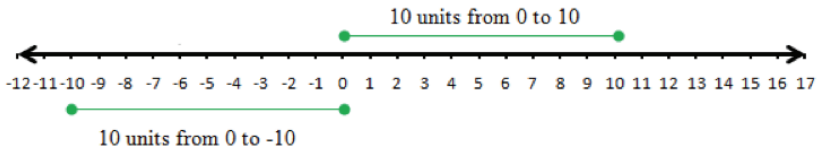
Activity 5

State the set of all real numbers whose number of units from zero, on number line, are

1. greater than 4
2. less than 6

Hint Draw a number line

Recall that absolute value of a number is the number of units from zero to a number line. That is, $|x| = k$ means k units from zero (k is a positive real number or zero)



For all real number x and $k \geq 0$

a) $|x| < k \Leftrightarrow -k < x < k$

b) $|x-a| < k \Leftrightarrow a-k < x < a+k$

c) $|x| > k \Leftrightarrow x > k \ \ or \ \ x < -k$

d) $|x-a| > k \Leftrightarrow x > a+k \ \ or \ \ x < a-k$

Example 5

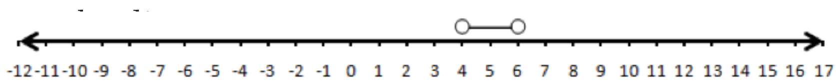
Find the solution set of the inequality $|3x-15| < 3$

Solution

$|3x-15| < 3 \Leftrightarrow -3 < 3x-15 < 3$

$-3+15 < 3x < 3+15 \Leftrightarrow 12 < 3x < 18$ or $\dfrac{12}{3} < x < \dfrac{18}{3} \Leftrightarrow 4 < x < 6$

Solution set is $S = \{x \in IR : 4 < x < 6\}$



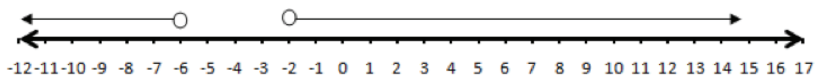
Example 6

Solve the inequality $|x+4| > 2$

Solution

$$|x+4| > 2 \Leftrightarrow x+4 > 2 \ \ or \ \ x+4 < -2$$
$$\Leftrightarrow x > 2-4 \ \ or \ \ x < -2-4$$
$$\Leftrightarrow x > -2 \ \ or \, x < -6$$

This is the set



Example 7

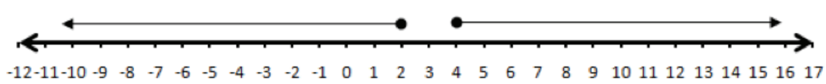
Find the solution set of the inequality $|x-3|\geq 1$

Solution

$|x-3|\geq 1 \Leftrightarrow x-3\geq 1 \qquad x-3\leq -1$

$\Leftrightarrow x\geq 4 \qquad x\leq 2$

Number line:



Example 8

A technician measures an electric current which is 0.036 A with a possible error of ±0.002 A. Write this current, i, as an inequality with absolute values.

Solution

The possible error of ±0.002 A means that the difference between the actual current and the value of 0.036 A cannot be more than 0.002 A.

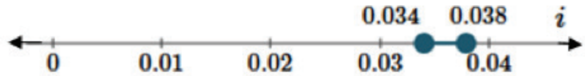
So the values of i we have can be expressed as:

$0.034\leq i\leq 0.038$

We can simply write this as:

$|i-0.036|\leq 0.002$

Number line:



Exercise 5

Solve the following inequalities:

1) $|2x-1|>5$ 2) $2\left|\frac{2x}{3}+1\right|\geq 4$ 3) $|3-2x|<3$

Equations and inequalities in real life problems

Activity 6

How can you do the following?

1. A father is 30 years older than his son. 5 years ago he was four times as old as his son. What is the son's age?
2. Betty spent one fifth of her money on food. Then she spent half of what was left for a haircut. She bought a present for 7,000 francs. When she got home, she had 13,000 francs left. How much did Betty have originally?

Equations can be used to solve real life problems.

To solve real life problems, follow the following steps:

a) Identify the variable and assign symbol to it.

b) Write down the equation.

c) Solve the equation.

d) Interpret the result. There may be some restrictions on the variable.

Example 9

Kalisa is four times as old as his son, and his daughter is 5 years younger than his brother. If their combined ages amount to 73 years, find the age of each person.

Solution

Let x stands for the age of the son. Then $4x$ is the age of Kalisa and $x-5$ is the age of daughter. So that $x+4x+x-5=73$.

$6x=78 \Rightarrow x=13$

The age of the son is $x=13$ years old.

The age of Kalisa is $4x=52$ years old.

The age of daughter is $x-5=8$ years old.

Example 10

Concrete is a mixture of cement, sand and aggregate. If 4kg of sand and 6kg of aggregate are used with each kg of cement, how many kg of each are required to make 1,210 kg?

Solution

Let x be the amount of cement. Then, the amount of sand is $4x$ and the amount of aggregate is $6x$

$x+4x+6x=1{,}210$

$\Leftrightarrow 11x=1210$

$$\Rightarrow x=\frac{1{,}210}{11}=110$$

Thus,

110 kg of cement are required.

440 kg of sand are required.

660 kg of aggregate are required.

Example 11

John has 1,260,000 Francs in an account with his bank. If he deposits 30,000 Francs each week into the account, how many weeks will he need to have more than 1,820,000 Francs on his account?

Solution

Let x be the number of weeks

We have;

total amount of deposits to be made + the current balance > total amount wanted.

That is;

$30,000x + 1,260,000 > 1,820,000$

$30,000x > 1,820,000 - 1,260,000$

$30,000x > 560,000$

$$x > \frac{560,000}{30,000} \approx 19$$

Thus, he needs atleast 19 weeks.

Example 12

The yield from 50 acres of wheat was more than 1,250 tones. What was the yield per acre?

Solution

Let m be the yield.

Total number of yield > number of tones.

That is;

$50m > 1,250$

$$m > \frac{1,250}{50}$$

$m > 25$

The yield per acre was more than 25 tones.

Exercise 6

1. The sum of two numbers is 25. One of the numbers exceeds the other by 9. Find the numbers.
2. The difference between the two numbers is 48. The ratio of the two numbers is 7:3. What are the two numbers?
3. The length of a rectangle is twice its breadth. If the perimeter is 72 metre, find the length and breadth of the rectangle.

4. Aaron is 5 years younger than Ron. Four years later, Ron will be twice as old as Aaron. Find their present ages.
5. Sam and Alex play in the same soccer team. Last Saturday Alex scored 3 more goals than Sam, but together they scored less than 9 goals. What are the possible number of goals Alex scored?
6. Joe enters a race where he has to cycle and run. He cycles a distance of 25 km, and then runs for 20 km. His average running speed is half of his average cycling speed. Joe completes the race in less than 2½ hours, what can we say about his average speeds?

2. Simultaneous equations in two unknowns

Consider the following system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

If $c_1 = c_2 = 0$, the system becomes $\begin{cases} a_1x + b_1y = 0 \\ a_2x + b_2y = 0 \end{cases}$ and it is said to be homogeneous system.

Algebraically, there are three methods for solving this system: combination method, substitution method and Cramer's rule.

Note

- The solution $\frac{b}{0}, b \neq 0$ means **impossible**
- The solution $\frac{0}{0}$ means **indeterminate**

Combination (or addition or elimination) method

Activity 7

For each of the following, find two numbers to be multiplied to the equations such that one variable will be eliminated;

1. $\begin{cases} x + y = 12 \\ 2x + y = 4 \end{cases}$
2. $\begin{cases} 3x - y = 20 \\ -x + 2y = 4 \end{cases}$
3. $\begin{cases} x - 2y = 10 \\ 2x + y = 14 \end{cases}$

We try to combine the two equations such that we will remain with one equation with one unknown. We find two numbers to be multiplied on each equation and then add up such that one unknown is cancelled.

Example 13

Solve the following system:

$$\begin{cases} x+y=1 \\ 2x+3y=2 \end{cases}$$

Solution

$$\begin{cases} x+y=1 \\ 2x+3y=2 \end{cases} \begin{vmatrix} -2 \\ 1 \end{vmatrix} \Leftrightarrow \begin{cases} -2x-2y=-2 \\ 2x+3y=2 \end{cases}$$

$$y=0$$

$x+y=1 \Leftrightarrow x=1-y=1$

$S=\{(1,0)\}$

Example 14

Solve the following system:

$$\begin{cases} 3x-2y=5 \\ y+4x=1 \end{cases}$$

Solution

$$\begin{cases} 3x-2y=5 \\ y+4x=1 \end{cases} \begin{vmatrix} 1 \\ 2 \end{vmatrix} \Leftrightarrow \begin{cases} 3x-2y=5 \\ 2y+8x=2 \end{cases}$$

$$11x=7 \Rightarrow x=\frac{7}{11}$$

$$y+4x=1 \Leftrightarrow y=1-4x=1-\frac{28}{11}=\frac{11-28}{11}=-\frac{17}{11}$$

$$S=\left\{\left(\frac{7}{11},-\frac{17}{11}\right)\right\}$$

Example 15

Solve the following system:

$$\begin{cases} y-2x=2 \\ 2y-4x=-3 \end{cases}$$

Solution

$$\begin{cases} y-2x=2 \\ 2y-4x=-3 \end{cases} \begin{vmatrix} -2 \\ 1 \end{vmatrix} \Leftrightarrow \begin{cases} -2y+4x=-4 \\ 2y-4x=-3 \end{cases}$$

$$0x=-7 \quad \textit{impossible}$$

No solution

Example 16

Solve the following system:

$$\begin{cases} x+y=2 \\ 2x+2y=4 \end{cases}$$

Solution

$$\begin{cases} x+y=2 \\ 2x+2y=4 \end{cases}\begin{vmatrix} -2 \\ 1 \end{vmatrix} \Leftrightarrow \begin{cases} -2x-2y=-4 \\ 2x+2y=4 \end{cases}$$

$$0x=0 \quad \textit{indeterminate}$$

There is infinite number of solutions.

Exercise 7

Use elimination method to solve;

1. $\begin{cases} x-y=3 \\ 2x-2y=6 \end{cases}$
2. $\begin{cases} -x+4y=0 \\ 2x-7y=0 \end{cases}$
3. $\begin{cases} -3y+4x=10 \\ x+3y=5 \end{cases}$
4. $\begin{cases} 5y+3x=9 \\ 10x+6y=10 \end{cases}$
5. $\begin{cases} 3x-4y=1 \\ x-3y=2 \end{cases}$
6. $\begin{cases} x-4y=1 \\ x-y=2 \end{cases}$

Substitution method

Activity 8

In each of the following systems find the value of one variable from one equation and substitute it in the second.

1. $\begin{cases} x-y=5 \\ x+2y=6 \end{cases}$
2. $\begin{cases} x+2y=10 \\ -3x+2y=12 \end{cases}$
3. $\begin{cases} x+y=-10 \\ 4x+y=0 \end{cases}$

We find the value of one unknown in one equation and put it in another equation to find the value of the remaining unknown.

Example 17

Solve the following system:

$$\begin{cases} x+y=1 \\ 2x+3y=2 \end{cases}$$

solution

From the first equation, $x=1-y$. Put this value in second equation:

$$2(1-y)+3y=2 \Leftrightarrow 2-2y+3y=2$$
$$\Leftrightarrow y=2-2 \Rightarrow y=0$$
$$x=1-y \Rightarrow x=1-0=1$$
$$S=\{(1,0)\}$$

Example 18

Solve the following system:

$$\begin{cases} 3x-2y=5 & (1) \\ y+4x=1 & (2) \end{cases}$$

solution

From (2): $y=1-4x \quad (3)$

(3) in (1): $3x-2(1-4x)=5 \Leftrightarrow 3x-2+8x=5$
$\Leftrightarrow 11x=7 \Rightarrow x=\frac{7}{11}$

$$y=1-4x=1-4\left(\frac{7}{11}\right)=\frac{11-28}{11}=-\frac{17}{11}$$

$$S=\left\{\left(\frac{7}{11},-\frac{17}{11}\right)\right\}$$

Example 19

Solve the following system:

$$\begin{cases} y-2x=2 \\ 2y-4x=-3 \end{cases}$$

solution

$y=2+2x$' $2(2+2x)-4x=-3$
$\Leftrightarrow 4+4x-4x=-3$, $0x=-7$ impossible

This system is inconsistent. Thus, there is no solution.

Example 20

Solve the following system:

$$\begin{cases} x + y = 2 \\ 2x + 2y = 4 \end{cases}$$

$$x = 2 - y\text{'}\ \ 2(2-y) + 2y = 4$$

$$\Leftrightarrow 4 - 2y + 2y = 4$$

$$\Leftrightarrow 0y = 0 \Rightarrow y = \frac{0}{0}\ \ indeterminate$$

This system is a dependent system. Thus, there is an infinity solutions.

Exercise 8

Use elimination method to solve;

1. $\begin{cases} x - y = -4 \\ 3x - 3y = -12 \end{cases}$
2. $\begin{cases} -x + y = 0 \\ x - 7y = 0 \end{cases}$
3. $\begin{cases} -2y + 4x = 15 \\ x + y = 6 \end{cases}$
4. $\begin{cases} 5y - 3x = 9 \\ 10x - 6y = 10 \end{cases}$
5. $\begin{cases} 3x - 2y = 1 \\ 2x - 3y = 0 \end{cases}$
6. $\begin{cases} x - 2y = 1 \\ 2x + y = 2 \end{cases}$

Cramer's rule (determinants method)

Activity 9

Find the following determinants.

Hint: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$

1. $\begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix}$
2. $\begin{vmatrix} 3 & 7 \\ -2 & 1 \end{vmatrix}$
3. $\begin{vmatrix} 10 & -1 \\ -5 & 3 \end{vmatrix}$

In order to use Cramer's rule, x's must be in the same position and y's in the same position.

Consider the following system

$$\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases}$$

To use Cramer's rule, first we find;

$$\Delta=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=a_1b_2-a_2b_1 \ ,$$

$$\Delta_x=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}=c_1b_2-c_2b_1 \quad \text{and}$$

$$\Delta_y=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}=a_1c_2-a_2c_1 \qquad \text{Then} \quad x=\frac{\Delta_x}{\Delta},\ y=\frac{\Delta_y}{\Delta}$$

Example 21

Solve the following system:

$$\begin{cases} x+y=1 \\ 2x+3y=2 \end{cases}$$

solution

$$\Delta=\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}=(1)(3)-(2)(1)=1$$

$$\Delta_x=\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}=(1)(3)-(2)(1)=1$$

$$\Delta_y=\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}=(1)(2)-(2)(1)=0$$

$$x=\frac{\Delta_x}{\Delta}=1,\ y=\frac{\Delta_y}{\Delta}=0$$

$$S=\{(1,0)\}$$

Example 22

Solve the following system:

$$\begin{cases} 3x-2y=5 \\ y+4x=1 \end{cases}$$

solution

First rearrange the system such that x's will be in the same position and y's will be in the same position.

$$\begin{cases}3x-2y=5\\4x+y=1\end{cases}$$

$$\Delta=\begin{vmatrix}3 & -2\\4 & 1\end{vmatrix}=(3)(1)-(4)(-2)=11$$

$$\Delta_x=\begin{vmatrix}5 & -2\\1 & 1\end{vmatrix}=(5)(1)-(1)(-2)=7$$

$$\Delta_y=\begin{vmatrix}3 & 5\\4 & 1\end{vmatrix}=(3)(1)-(4)(5)=-17$$

$$x=\frac{\Delta_x}{\Delta}=\frac{7}{11},\ y=\frac{\Delta_y}{\Delta}=-\frac{17}{11}$$

$$S=\left\{\left(\frac{7}{11},-\frac{17}{11}\right)\right\}$$

Example 23

Solve the following system:

$$\begin{cases}y-2x=2\\2y-4x=-3\end{cases}$$

solution

$$\begin{cases}y-2x=2\\2y-4x=-3\end{cases}\Leftrightarrow\begin{cases}-2x+y=2\\-4x+2y=-3\end{cases}$$

$$\Delta=\begin{vmatrix}-2 & 1\\-4 & 2\end{vmatrix}=(-2)(2)-(-4)(1)=0$$

$$\Delta_x=\begin{vmatrix}2 & 1\\-3 & 2\end{vmatrix}=(2)(2)-(-3)(1)=7$$

$$\Delta_y=\begin{vmatrix}-2 & 2\\-4 & -3\end{vmatrix}=(-2)(-3)-(-4)(2)=14$$

$$x=\frac{\Delta_x}{\Delta}=\frac{7}{0},\ y=\frac{\Delta_y}{\Delta}=\frac{14}{0}$$

This system is inconsistent. Thus, there is no solution.

Example 24

$$\begin{cases} x+y=2 \\ 2x+2y=4 \end{cases}$$

solution

Solve the following system:

$$\Delta = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = (1)(2)-(2)(1)=0$$

$$\Delta_x = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = (2)(2)-(4)(1)=0$$

$$\Delta_y = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = (1)(4)-(2)(2)=0$$

$$x=\frac{\Delta_x}{\Delta}=\frac{0}{0},\ y=\frac{\Delta_y}{\Delta}=\frac{0}{0}$$

This system is a dependent system. Thus, there is an infinity solutions.

Exercise 9

Use Cramer's rule to solve;

1. $\begin{cases} x+y=2 \\ 4x-4y=8 \end{cases}$
2. $\begin{cases} -x+y=0 \\ x+2y=3 \end{cases}$
3. $\begin{cases} -y+4x=8 \\ x+2y=3 \end{cases}$
4. $\begin{cases} 5y+3x=2 \\ 10x+6y=0 \end{cases}$
5. $\begin{cases} 3x+3y=1 \\ 2x-3y=4 \end{cases}$
6. $\begin{cases} 3x-5y=10 \\ 2x+y=12 \end{cases}$

Graphical method

Activity 10

Given the system

$$\begin{cases} 3x+y=10 \\ x-y=2 \end{cases}$$

1. For each equation, choose any two values of x and use them to find values of y; this gives you two points in the form (x,y).

2. Plot the obtained points in xy plane and join these points to obtain the lines. Two points for each equation give one line.
3. What is the point of intersection for two lines?

Some systems of linear equations can be solved graphically. To do this, follow the following steps:

1. Find at least two points for each equation.
2. Plot the obtained points in xy plane and join these points to obtain the lines. Two points for each equation give one line.
3. The point of intersection for two lines is the solution for the given system

Example 25

Solve the following system by graphical method

$$\begin{cases} x+y=4 \\ x-y=2 \end{cases}$$

Solution

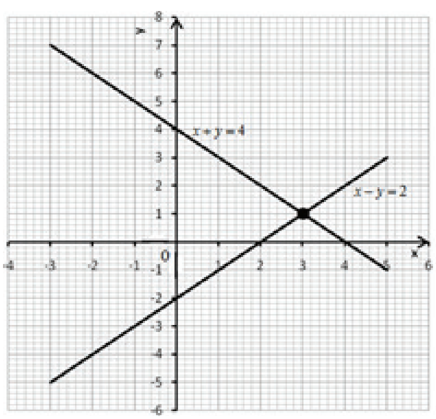
For $x+y=4$

x	-3	5
y	7	-1

For $x-y=2$

x	-3	5
y	−5	3

Graph



The two lines intersect at point (3,1). Therefore the solution is $S=\{(3,1)\}$.

Example 26

Solve the following equations graphically if possible

$$\begin{cases} y-2x=2 \\ 2y=4x-3 \end{cases}$$

Solution

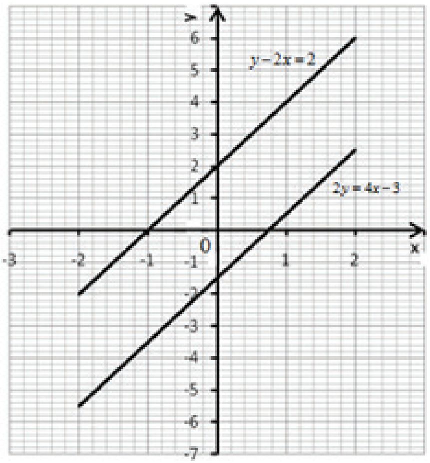
For $y-2x=2$

x	-2	2
y	−2	6

For $2y=4x-3$

x	-2	2
y	−5.5	2.5

Graph



We see that the two lines are parallel and do not intersect. Therefore there is no solution. Note that the gradients of the two lines are the same.

Example 27

Solve the following equations graphically if possible

$$\begin{cases} x+y=2 \\ 2y=4-2x \end{cases}$$

Solution

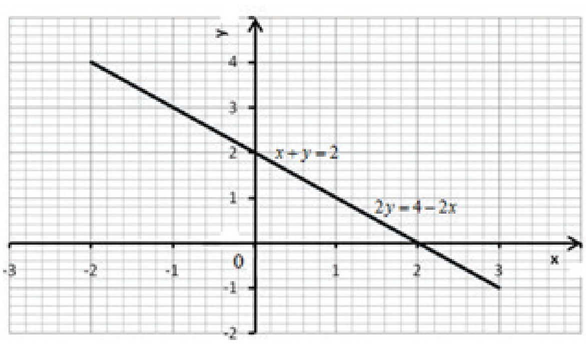
For $x+y=2$

x	-2	3
y	4	-1

For $2y=4-2x$

x	-2	3
y	4	-1

Graph



We see that the two lines coincide as a single line. In such case there is infinite number of solutions.

Exercise 10

Solve the following system by graphical method

1. $\begin{cases} 3x+y=1 \\ x-y=-1 \end{cases}$ 2. $\begin{cases} 2x+y=3 \\ 6x+3y=9 \end{cases}$ 3. $\begin{cases} 2x-y=3 \\ 6x-3y=0 \end{cases}$

Solving word problems using simultaneous equations

Activity 11

How can you do the following question?

Margie is responsible for buying a week's supply of food and medication for the dogs and cats at a local shelter. The food and medication for each dog costs twice as much as those supplies for a cat. She needs to feed 164 cats and 24 dogs. Her budget is $4240. How much can Margie spend on each dog for food and medication?

To solve word problems, follow the following steps:

a) Identify the variables and assign symbol to them.
b) Express all the relationships, among the variables using equations.
c) Solve the simultaneous equations.
d) Interpret the result. There may be some restrictions on the variables.

Example 28

Peter has 23 coins in his pocket. Some of them are 5 Frw coins and the rest are 10 Frw coins. The total value of coins is 205 Frw . Find the number of 10 Frw coins and the number of 5 Frw coins.

Solution

Let x be the number of 10 Frw coins and y be the number of 5 Frw coins. Then,

$$\begin{cases} x+y=23 \\ 10x+5y=205 \end{cases}$$

$$\Leftrightarrow \begin{cases} x+y=23 \\ 2x+y=41 \end{cases}$$

From first equation, $x=23-y$.

In second equation,

$$2(23-y)+y=41$$

$$46-2y+y=41$$

$$\Leftrightarrow -y=-5 \Rightarrow y=5 \quad and \quad x=23-5=18$$

Thus, there are 18 coins of 10 Frw and 5 coins of 5 Frw .

Example 29

Cinema tickets for 2 adults and 3 children cost 1,200 Frw . The cost for 3 adults and 5 children is 1,900 Frw. Find the cost of an adult ticket and the cost of a child ticket.

Solution

Let x be the cost of an adult ticket and y be the cost of a child ticket, then

$$\begin{cases} 2x+3y=1200 \\ 3x+5y=1900 \end{cases}$$

Solve for x and y

From first equation: $2x=1200-3y \Rightarrow x=\frac{1200-3y}{2}$

Put this value in second equation;

$$3\left(\frac{1200-3y}{2}\right)+5y=1900$$

$$\Leftrightarrow \frac{3600-9y}{2}+5y=1900$$

$$\Leftrightarrow 3600-9y+10y=3800$$

$$\Rightarrow y=200$$

And $x=\frac{1200-600}{2}=300$

Thus, the cost of an adult ticket is 300 Frw and the cost of a child ticket is 200 Frw

Exercise 11

1. A test has twenty questions worth 100 points. The test consists of True/False questions worth 3 points each and multiple choice questions worth 11 points each. How many multiple choice questions are on the test?
2. Two small pitchers and one large pitcher can hold 8 cups of water. One large pitcher minus one small pitcher constitutes 2 cups of water. How many cups of water can each pitcher hold?
3. The state fair is a popular field trip destination. This year, the senior class at High School A and the senior class at High School B both planned trips there. The senior class at High School A rented and filled 8 vans and 8 buses with 240 students. High School B rented and filled 4 vans and 1 bus with 54 students. Every van had the same number of students in it as did the buses. Find the number of students in each van and in each bus.
4. The sum of the digits of a certain two-digit number is 7. Reversing its digits increases the number by 9. What is the number?
5. A boat traveled 210 miles downstream and back. The trip downstream took 10 hours. The trip back took 70 hours. What is the speed of the boat in still water? What is the speed of the current?

3. Quadratic equations and inequalities

Equations of the type $ax^2+bx+c=0$ $(a \neq 0)$ are called quadratic equations.

There are three main ways of solving such equations:

a) By factorizing or finding square roots

b) By the formula $x=\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

c) By completing the square

Quadratic equations by factorizing or finding square roots

Activity 12

Smoke jumpers are firefighters who parachute into areas near forest fires. Jumpers are in free fall from the time they jump from a plane until they open their parachutes. The function $y=-16t^2+1600$ gives a jumper's height y in metre after t seconds for a jump from $1600m$. How long is free fall if the parachute opens at $1000m$?

The method of solving quadratic equations by factorization should only be used if is readily factorized by inspection.
The method of solving quadratic equations by factorization should only be used if is readily factorized by inspection.

Example 30

Solve in $\mathbb{R}$: $x^2+2x-24=0$

Solution

$$x^2+2x-24=0 \Leftrightarrow (x+6)(x-4)=0$$

So, either $x+6=0$ or $x-4=0$ giving $x=-6$ or $x=4$.

Example 31

Solve in $\mathbb{R}$: $5x^2+7x-6=0$

Solution

$$5x^2+7x-6=0 \Leftrightarrow 5x^2-3x+10x-6=0$$

$$\Leftrightarrow x(5x-3)+2(5x-3)=0$$

$$\Leftrightarrow (5x-3)(x+2)=0$$

So, either $5x-3=0$ or $x+2=0$ giving $x=\frac{3}{5}$ or $x=-2$.

Exercise 12

Solve in set of real numbers the following equations by factorization

1. $x^2+6x+8=0$
2. $x^2-2x=3$
3. $2x^2+6x=-4$
4. $12x^2-154=0$

Quadratic equations by completing the square

Activity 13

a) By completing the square, show that $y=ax^2+bx+c$ can be written as

$$y=a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right) \text{ if } a\neq 0$$

b) Use the result in a) to solve

$2x^2-7x-4=0$

Before solving quadratic equations by completing the square, let's look at some examples of expanding a binomial by squaring it.

$(x+3)^2=x^2+6x+9$.

$(x-5)^2=x^2-10x+25$

Notice that the constant term (k^2) of the trinomial is the square of half of the coefficient of trinomial's x-term. Thus, to make the expression x^2+kx a perfect square, you must add $\left(\frac{1}{2}k\right)^2$ to the expression.

When completing the square to solve quadratic equation, remember that you must preserve the equality. When you add a constant to one side of the equation, be sure to add the same constant to the other side of equation.

Example 32

Solve $x^2 - 4x + 1 = 0$ by completing the square

Solution

$x^2 - 4x + 1 = 0$	Rewrite original equation
$x^2 - 4x = 1$	Subtract 1 from both sides.
$x^2 - 4x + (-2)^2 = 1 + (-2)^2$	Add $(-2)^2 = 4$ to both sides.
$(x-2)^2 = 5$	Binomial squared.
$x - 2 = \pm\sqrt{5}$	Take square roots.
$x = 2 \pm \sqrt{5}$	Solve for x.

The equation has two solutions: $x = 2 + \sqrt{5}$ and $x = 2 - \sqrt{5}$

Example 33

Solve $4x^2 + 2x - 5 = 0$ by completing the square

Solution

$4x^2 + 2x - 5 = 0$	Rewrite original equation
$4x^2 + 2x = 5$	Add 5 to both sides.
$x^2 + \frac{1}{2}x = \frac{5}{4}$	Divide both sides by 4.
$x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)^2 = \frac{5}{4} + \frac{1}{16}$	Add $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$ to both sides.

$\left(x+\frac{1}{4}\right)^2=\frac{21}{16}$ Binomial squared.

$x+\frac{1}{4}=\pm\frac{\sqrt{21}}{2}$ Take square roots.

$x=-\frac{1}{4}\pm\frac{\sqrt{21}}{2}$ Solve x.

The equation has two solutions: $x=-\frac{1}{4}+\frac{\sqrt{21}}{2}$ and $x=-\frac{1}{4}-\frac{\sqrt{21}}{2}$

Exercise 13

Solve in set of real numbers the following equations by completing the square

1. $x^2+5x-24=0$
2. $x^2-13x+36=0$
3. $2x^2-x-6=0$
4. $3x^2+5x-12=0$

Quadratic equations by the formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Activity 14

Think about two numbers a and b (a can be equal to b) such that

1. a+b=4 and ab =4
2. a+b=5 and ab=6
3. a+b=7 and ab =12
4. a+b=$\frac{3}{2}$ and ab =$\frac{1}{2}$
5. a+b=–2 and ab =–35

Let x and y be two real numbers such that $x+y=s$ and $xy=p$. s is the sum and p is the product of two roots.

Here $y = s - x$ and $x(s-x) = p$. Or $sx - x^2 = p$ or $x^2 - sx + p = 0$. This equation is said to be quadratic equation and s, p are the sum and product of the two roots respectively.

Quadratic equation or equation of second degree has the form where the sum of two roots is $s = -\frac{b}{a}$ and their product is $p = \frac{c}{a}$. To solve this equation, first we find the discriminant (delta): $\Delta = b^2 - 4ac$

In fact,

$$ax^2 + bx + c = 0$$

$$\Leftrightarrow ax^2 + bx = -c$$

$$\Leftrightarrow a\left(x^2 + \frac{b}{a}x\right) = -c \;\; as \;\; a \neq 0$$

$$\Leftrightarrow x^2 + \frac{b}{a}x = -\frac{c}{a} \;\; as \;\; a \neq 0 \text{ (making the coefficient of } x^2 \text{ one)}$$

$$\Leftrightarrow \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} \leftarrow \frac{b^2}{4a^2}$$

is the square of half the coefficient of x, $\left(\frac{b}{a}\right)$, *in* $x^2 + \frac{b}{a}x$

$$\Leftrightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b \;\; - \;\; ac}{4a}}$$

$$\Leftrightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b \;\; - \;\; ac}{4a}}$$

$$\Leftrightarrow x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2|a|}$$

$$\Leftrightarrow x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}, if\ a>0$$

$$or\ x+\frac{b}{2a}=\mp\frac{\sqrt{b^2-4ac}}{2a}, if\ a<0$$

Simply,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Let $\Delta=b^2-4ac$

There are three cases:

- If $\Delta>0$, there are two distinct real roots:

  $x_1=\frac{-b+\sqrt{\Delta}}{2a}$ and $x_2=\frac{-b-\sqrt{\Delta}}{2a}$

- If $\Delta=0$, there is one repeated real root (one double root):

  $$x_1=x_2=\frac{-b}{2a}$$

- If $\Delta<0$, there is no real root.

Example 34

Solve in $\mathbb{R}$ *:* $x^2+2x+1=0$

Solution

$$x^2+2x+1=0$$

$$\Delta=2^2-4(1)(1)=0$$

$$x_1=x_2=\frac{-2}{2}=-1$$

$$S=\{-1,-1\}$$

Example 35

Solve in $\mathbb{R}: x^2 - 7x + 5 = -5$

Solution

$x^2 - 7x + 5 = -5$

$\Leftrightarrow x^2 - 7x + 10 = 0$

As we saw it, in this equation the sum of two roots is 7 and the product is 10. To find those roots we can think about two numbers such that their sum is 7 and their product is 10. Those numbers are 2 and 5. Thus $S = \{2, 5\}$

Or

$\Delta = (-7)^2 - 4(1)(10) = 9$

$$x_1 = \frac{-(-7) + \sqrt{9}}{2} = 5,\ x_1 = \frac{-(-7) - \sqrt{9}}{2} = 2$$

$S = \{2, 5\}$

Example 36

Solve in $\mathbb{R}: 2x^2 + 3x + 4 = 0$

Solution

$2x^2 + 3x + 4 = 0$

$\Delta = 3^2 - 4(2)(4) = -23$

Since $\Delta < 0$, there is no real root.

Then, $S = \varnothing$

Example 37

For what value of k will the equation $x^2 + 2x + k = 0$ have one double roots? Find that root.

Solution

For one double root $\Delta = 0$.

$\Delta = 4 - 4k$

$4 - 4k = 0 \Rightarrow k = 1$

Thus, the value of k is 1.

That root is $x = -\frac{2}{2} = -1$.

Exercise 14

Solve in set of real numbers;

1. $x^2 - 12x + 11 = 0$
2. $x^2 + 2x = 35$
3. $x^2 - 3x = -11$
4. $3x^2 - 7x + 2 = 0$
5. $x^2 - 121 = 0$

Notice:

This method of solving quadratic equations by the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ helps us to write down factor form of a quadratic expression.

Factor form of a quadratic expression

Activity 15

In each of the following, remove brackets and discuss about the result and original form.

1. $(x+4)(x-1)$
2. $3(x-2)(x-5)$
3. $(x+2)(x+1)$
4. $6(x-3)(x-8)$
5. $(x-6)(x+2)$

A quadratic expression $ax^2 + bx + c$ can be written in factor form. To do this, solve for $ax^2 + bx + c = 0$ to find the 2 real roots.

If the two real roots exist, i.e $\Delta \geq 0$, then the factor form is $ax^2 + bx + c = a(x - x_1)(x - x_2)$. If there is no real root, i.e

$\Delta < 0$, then the expression $ax^2 + bx + c$ has no factor form.

If there is no real root, i.e , then the expression has no factor form.

Example 38

Put the following expression in factor form $x^2 - 5x + 6$

Solution

$x^2 - 5x + 6 = 0$

$\Delta = (-5)^2 - 4(1)(6) = 1$

$x_1 = \frac{5+1}{2} = 3,\ x_2 = \frac{5-1}{2} = 2$

The factor form is

$x^2 - 5x + 6 = (x-3)(x-2)$

Example 39

Put the following expression in factor form $a^2 - 4a + 4$

Solution

$a^2 - 4a + 4$

$\Delta = 16 - 16 = 0$

$a_1 = a_2 = \frac{4}{2} = 2$

The factor form is

$a^2 - 4a + 4 = (a-2)(a-2)$

Example 40

Put the following expression in factor form $2t^2 + 3t + 10$

Solution

$2t^2 + 3t + 10 = 0$

$\Delta = 9 - 80 = -71$

Since $\Delta < 0$, the expression $2t^2 + 3t + 10$ has no factor form.

Exercise 15

Find the factor form of

1. $x^2 - 10x + 16$
2. $x^2 - x + 1$
3. $6x^2 - 5x + 1$
4. $x^2 + 4x - 5$
5. $4x^2 + 7x - 2$

Equations Reducible to Quadratic Form

a) Biquadratic equations

Activity 16

In each of the following rewrite the given equation letting

1. $x^4 - 2x^2 + 2 = 0$ 2. $6x^4 + 5x^2 + 1 = 0$

3. $x^4 - 13x^2 + 36 = 0$

Biquadratic equations are the equations that if we look at them in the correct light we can make them look like quadratic equations. A biquadratic equation has the form

$ax^{2n} + bx^n + c = 0$

To solve a biquadratic equations, change $x^2 = y$. This generates a quadratic equation with the unknown, y:

$ay^2 + by + c = 0$

For every positive value of y there are **two values of x**, find: $x = \pm\sqrt{y}$

Example 41

Solve $x^4 - 7x^2 + 12 = 0$

Solution

Here $x^4 = \left(x^2\right)^2$

Let $y = x^2$, then $y^2 = x^4$

Now, the given equation becomes $y^2 - 7y + 12 = 0$

$y^2 - 7y + 12 = (y-3)(y-4) = 0 \Rightarrow y = 3$ or $y = 4$

But $y = x^2$

$y = 3: \ 3 = x^2 \Rightarrow x = \pm\sqrt{3}$

$y = 4: \ 4 = x^2 \Rightarrow x = \pm 2$

So, we have four solutions to the original equation, $x = \pm 2$ and $\pm\sqrt{3}$.

So, the basic process is to check that the equation is reducible to a quadratic form, then make a quick substitution to turn it into a quadratic equation. In most cases, to make the check that it's reducible to quadratic form, all we really need to do is to check that one of the exponents is twice the other.

Exercise 16

Solve in set of real numbers

1. $x^4 - 13x^2 + 36 = 0$
2. $x^6 - 7x^3 + 6 = 0$
3. $x^4 - 10x^2 + 9 = 0$
4. $x^4 - 61x^2 + 900 = 0$

b) Nested radicals

Activity 17

Let $\sqrt{4+\sqrt{12}} = \sqrt{x} + \sqrt{y}$

By squaring both sides find the values of x and y.

A **nested radical** is a radical expression (one containing a square root sign, cube root sign, etc) that contains (nests) another radical expression. Examples include $\sqrt{5-2\sqrt{5}}$ and more complicated ones such as $\sqrt[3]{2+\sqrt{3}+\sqrt[3]{4}}$.

We will see the nested radicals of the form $\sqrt{A \pm \sqrt{B}}$

The radical like of $\sqrt{A \pm \sqrt{B}}$ can be transformed and give $\sqrt{x} \pm \sqrt{y}$.

The process is called **denesting**.

To do this we square both side of the relation
$\sqrt{A \pm \sqrt{B}} = \sqrt{x} \pm \sqrt{y}$ and we find the values of x and y.
That is,

$$\sqrt{A \pm \sqrt{B}} = \sqrt{x} \pm \sqrt{y} \Leftrightarrow A \pm \sqrt{B} = \left(\sqrt{x} \pm \sqrt{y}\right)^2$$

$$\Leftrightarrow A \pm \sqrt{B} = x \pm 2\sqrt{xy} + y$$

$$\Leftrightarrow A \pm \sqrt{B} = x + y \pm \sqrt{4xy}$$

$$\Leftrightarrow \begin{cases} A = x + y \\ \dfrac{B}{4} = xy \end{cases}$$

Example 42

Transform the radical $\sqrt{9 + \sqrt{80}}$ to simple radical

Solution

Let $\sqrt{9 + \sqrt{80}} = \sqrt{x} + \sqrt{y}$

$$\left(\sqrt{9 + \sqrt{80}}\right)^2 = \left(\sqrt{x} + \sqrt{y}\right)^2$$

$$\Leftrightarrow 9 + \sqrt{80} = x + 2\sqrt{xy} + y$$

$$\Leftrightarrow 9 + \sqrt{80} = x + y + \sqrt{4xy}$$

$$\begin{cases} x + y = 9 \\ 4xy = 80 \end{cases} \Rightarrow \begin{cases} x + y = 9 \\ xy = 20 \end{cases}$$

We need two numbers such that their sum is 9 and their product is 20

$\Rightarrow x = 4,\ y = 5$ or $x = 5,\ y = 4$ Thus, $\sqrt{9 + \sqrt{80}} = \sqrt{4} + \sqrt{5}$

Example 43

Transform the radical $\sqrt{3 - \sqrt{5}}$ to simple radical.

Solution

Let $\sqrt{3-\sqrt{5}}=\sqrt{x}-\sqrt{y}$

$$\left(\sqrt{3-\sqrt{5}}\right)^2=\left(\sqrt{x}-\sqrt{y}\right)^2$$

$$\Leftrightarrow 3-\sqrt{5}=x-2\sqrt{xy}+y$$

$$\Leftrightarrow 3-\sqrt{5}=x+y-\sqrt{4xy}$$

$$\begin{cases}x+y=3\\4xy=5\end{cases}\Rightarrow\begin{cases}x+y=3\\xy=\dfrac{5}{4}\end{cases}$$

We need two numbers such that their sum is 3 and their product is $\frac{5}{4}$

$$\Rightarrow x=\frac{5}{2},\ y=\frac{1}{2}\quad\text{or}\quad x=\frac{1}{2},\ y=\frac{5}{2}$$

Because of negative sign between $\sqrt{x}$ and $\sqrt{y}$, we take the values of x and y such that $\sqrt{x}>\sqrt{y}$ as $\sqrt{3-\sqrt{5}}>0$.

Then $\sqrt{3-\sqrt{5}}=\sqrt{\dfrac{5}{2}}-\sqrt{\dfrac{1}{2}}$

Exercise 17

Solve in set of real numbers the following equations

1. $\sqrt{6-2\sqrt{5}}$
2. $\sqrt{6-2\sqrt{5}}$
3. $\sqrt{5+2\sqrt{6}}$

c) Irrational equations

Activity 18

Consider the following equation

$\sqrt{x+8}=x+2$

1. Square both sides of the equation
2. Solve the obtained equation
3. Verify that the obtained solutions are solution of the original equation and then give the solution set of the original equation (given equation)

Irrational equation is the equation involving radical sign. We will see the case the radical sign is a **square root**.

To solve an irrational equation, follow these steps:

a) Isolate a radical in one of the two members and pass it to another member of the other terms which are also radical.

b) Square both members.

c) Solve the equation obtained.

d) Check if the solutions obtained verify the initial equation.

e) If the equation has several radicals, repeat the first two steps of the process to remove all of them.

Example 44

Solve in set of real numbers

$$1+\sqrt{x^2-9}=x$$

Solution

$$\sqrt{x^2-9}=x-1 \Leftrightarrow x^2-9=(x-1)^2$$

After developing and simplifying we find $x=5$. We test this value and we see that it is not false. The original equation has solution $x=5$

Thus, $S=\{5\}$

Example 45

Solve in set of real numbers

$$\sqrt{2x+8}+\sqrt{x+5}=7$$

Solution

$$\sqrt{2x+8}+\sqrt{x+5}=7 \Leftrightarrow \sqrt{2x+8}=7-\sqrt{x+5}$$

$$2x+8=\left(7-\sqrt{x+5}\right)^2$$

After developing and simplifying $x-46=-14\sqrt{x+5}$.
Squaring again, we obtain $x^2-288x+1136=0$

Either $x=4$ or $x=248$, but 4 is the only solution of the original equation.

Thus, $S=\{4\}$

Exercise 18

Solve in set of real numbers the following equations

1. $\sqrt{x+7}=13$ 2. $\sqrt{x-4}=-7$ 3. $2-\sqrt{x+3}=5$

Quadratic inequalities

Activity 19

Find the range where

1. $(x-2)(x+1)$ is positive 2. $(x-1)(x-2)$ is negative

We saw how to solve the inequality product like $(ax+b)(cx+d)>0$. If we find the product of the left hand side, the result will be a quadratic expression of the form (ax^2+bx+c).

Then to solve the inequality of the second degree like $ax^2+bx+c>0$ we need to put the expression ax^2+bx+c in factor form and use the method to solve inequality product. If the expression to be transformed in factor form has no factor form, we find its sign by replacing the unknown by any chosen real number. We may find that the expression is always positive or always negative.

If the expression to be transformed in factor form has a repeated root, it is zero at that root and positive or negative elsewhere depending on coefficient of x^2.

Example 46

$x^2-2x+1\le 0$

Solution

$x^2-2x+1=0$

$x_1=x_2=1$ and $x^2-2x+1=(x-1)(x-1)$

The expression x^2-2x+1 is zero for $x=1$ otherwise it is positive since $x^2-2x+1=(x-1)(x-1)=(x-1)^2$

The solution is only $x=1$ since we are given $x^2-2x+1\le 0$.

Example 47

$3x^2+x-14<0$

Solution

$3x^2+x-14=0$

$x_1=-\frac{7}{3}$ or $x_2=2$

$x^2+x-14=3\left(x+\frac{7}{3}\right)(x-2)=(3x+7)(x-2)$

x	$-\infty$		$-\frac{7}{3}$		2		$+\infty$
$3x+7$		-	0	+		+	
$x-2$		-		-	0	+	
$3x^2+x-14$		+	0	-	0	+	

Thus, $S=\left]-\frac{7}{3},2\right[$

Example 48

$x^2-4x+4\ge 0$

Solution

$x^2-4x+4=0$

$x_1=x_2=2$ and $x^2-4x+4=(x-2)(x-2)$ or

$x^2-4x+4=(x-2)^2$

The expression x^2-4x+4 is zero for $x=2$ and positive elsewhere. Then, S = IR

Example 49

$2x^2+2x\le 4x-10$

Solution

$2x^2+2x\le 4x-10$

$\Leftrightarrow 2x^2+2x-4x+10\le 0$

$\Leftrightarrow 2x^2-2x+10\le 0$

$\Leftrightarrow x^2-x+5\le 0$

$x^2-x+5=0$

$\Delta=1-20=-19$

The expression x^2-x+5 cannot be factorized. Let $x=0$

$0^2-0+5=5>0$. Then the expression x^2-x+5 is always positive and the solution set is an empty set.

Example 50

$-2x^2+2x-10<0$

Solution

$-2x^2+2x-10<0$

$x^2-x+5>0$

$x^2-x+5=0$

$\Delta=1-20=-19$

The expression x^2-x+5 cannot be factorized. Let $x=0$

$0^2-0+5=5>0$. Then the expression x^2-x+5 is always positive and the solution set is the set of real numbers.

Exercise 19

Solve in set of real numbers

1. $x^2-10x-20>0$
2. $6x^2-5x+1\le 0$
3. $x^2+2x+12>0$
4. $x^2+x+18<0$
5. $x^2-17x-72\le 0$

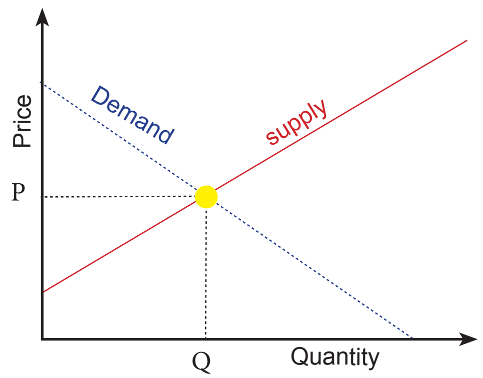
4. Applications

Activity 20

1. Explain how linear equations can be used in daily life1.
2. Give three examples of where you think quadratic equations are useful in daily life

a) Supply and demand analysis

Market equilibrium is when the amount of product produced is equal to the amount of quantity demanded. We can see equilibrium on a graph when the supply function and the demand function intersect, like shown on the graph below. Max can then figure out how to price his new lemonade products based on market equilibrium.



Example 51

Assume that in a competitive market the demand schedule is and the supply schedule is $p=60+0.4q$ (p=*price*, q=*quantity)*. If the market is in equilibrium then the equilibrium price and quantity will be where the demand and supply schedules intersect. As this will correspond to a point which is on both the demand schedule and the supply schedule the equilibrium values of p and q will be such that both equations hold.

To find the equilibrium quantity set $420-0.2q=60+0.4q$

$420-0.2q=60+0.4q$

$\Leftrightarrow 420-60=0.6q+0.2q$

$\Leftrightarrow 360=0.6q \Rightarrow q=600$

Then, $q=600$ is the equilibrium quantity.

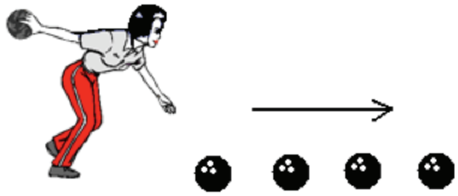
But, $p=420–0.2q$ then

$p=420–0.2(600)=300$

Thus, $p=300$ is the equilibrium price.

b) Linear motion

Linear motion is a motion along a straight line, and can therefore be described mathematically using only one spatial dimension. The linear motion can be of two types: uniform linear motion with constant velocity or zero acceleration; non uniform linear motion with variable velocity or non-zero acceleration.



Linear motion of the ball

Example 52

Some examples of linear motion are given below:

1. An athlete running 100m along a straight track
2. Parade of the soldiers
3. Car moving at constant speed
4. A bullet targeted from the pistol
5. A man swimming in the straight lane
6. Train moving in a straight track
7. Object dropped from a certain height
8. Balancing equation

c) Balancing equation

In chemistry, to balance the chemical equation we set the reactants and products equal to each other.

Example 53

To balance the chemical equation $Zn + HCl \longrightarrow ZnCl_2 + H_2$, we do the following:

There are two chlorines on the right but only one on the left; and the chlorine is in a single chemical species on each side. Put a 2 in front of the HCl on the left hand side.

$Zn + 2HCl \longrightarrow ZnCl_2 + H_2$

We see that the equation is now balanced, by comparing the numbers of atoms in the reactants and products, with one Zn on each side, two hydrogen on each side and two chlorines on each side.

d) Calculating Areas

People frequently need to calculate the area of things like rooms, boxes or plots of land.

Example 54

An example might involve building a rectangular box where one side must be twice the length of the other side. So long as this ratio is satisfied, the box will be as a person wants it. Say, however, that a person has only 4 square meters of wood to use for the bottom of the box. Knowing this, the person needs to create an equation for the area of the box using the ratio of the two sides. This means the area (the length times the width) in terms of x would equals , or . This equation must be less than or equal to 4 for the person to successfully make a box using his constraints. Once solved, he now knows that one side of the box must be   meters long, and the other must be   meters long.

e) Figuring out a profit

Sometimes calculating a business' profit requires using a quadratic function. If you want to sell something (even something as simple as lemonade) you need to decide how many things to produce so that you'll make a profit.

Example 55

Let us say that you're selling glasses of lemonade, and you want to make 12 glasses. You know, however, that you'll sell a different number of glasses depending on how you set your price. At 100 francs per glass, you are not likely to sell any, but at 10 francs per glass, you will probably sell 12 glasses in less than a minute. So, to decide where to set your price, use P as a variable. Let's say you estimate the demand for glasses of lemonade to be at 12 - P. Your revenue, therefore, will be the price times the number of glasses sold: P(12 - P), or

$12P - P^2$. Using however much your lemonade costs to produce, you can set this equation equal to that amount and choose a price from there.

f) Quadratics in Athletics

In athletic events that involve throwing things, quadratic equations are highly useful.

Example 56

Say, for example, you want to throw a ball into the air and have your friend catch it, but you want to give her the precise time it will take the ball to arrive.

To do this, you would use the velocity equation, which calculates the height of the ball based on a parabolic (quadratic) equation. So, say you begin by throwing the ball at 3 meters, where your hands are. Also assume that you can throw the ball upward at 14 meters per second, and that the earth's gravity is reducing the ball's speed at a rate of 5 meters per second squared. This means that we can calculate the height, using the variable t for time, in the form of $h=3+14t-t^2$. If your friend's hands are also at 3 metres in height, how many seconds will it take the ball to reach her? To answer this, set the equation equal to 3 = h, and solve for t. The answer is approximately 2.8 seconds.

g) Finding a Speed

Quadratic equations are also useful in calculating speeds. Avid kayakers, for example, use quadratic equations to estimate their speed when going up and down a river.

Example 57

Assume a kayaker is going up a river, and the river moves at 2 km/hr. Say he goes upstream -- against the current -- at 15 km, and the trip takes him 3 hours to go there and return. Remember that time = distance / speed. Let v = the kayak's speed relative to land, and let x = the kayak's speed in the water. So, we know that, while traveling upstream, the kayak's speed is v = x - 2 (subtract 2 for the resistance from the river current), and while going downstream, the kayak's speed is v = x + 2.

The total time is equal to 3 hours, which is equal to the time going upstream plus the time going downstream, and both distances are 15km. Using our equations, we know that 3 hours = $\frac{15}{(x-2)} + \frac{15}{(x+2)}$. Once this is expanded algebraically,

we get $3x^2 - 30x - 12 = 0$ or $x^2-10x-4=0$. Solving for x, we know that the kayaker moved his kayak at a speed of 10.39 km/hr

Unit summary

1. An equation is statement that the values of two mathematical expressions are equal while an inequality is a statement that the values of two mathematical that are not equal.
2. When we are given the equation $A \cdot B = 0$ then $A = 0$ or $B = 0$. Also $\frac{A}{B} = \frac{C}{D} \Leftrightarrow A \cdot D = B \cdot C$ where B, D ≠ O
3. To solve real life problems, follow the following steps:
   - Identify the variable and assign symbol to it
   - Write down the equation
   - Solve the equation
   - Interpret the result. There may be some restrictions on the variable.
4. Consider the following system

   $$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

   Algebraically, there are three methods for solving this system: combination method, substitution method and Cramer's rule. Some systems of linear equations can be solved graphically. To do this, follow the following steps:

   - Find at least two points for each equation.
   - Plot the obtained points in xy plane and join these points to obtain the lines. Two points for each equation give one line.
   - The point of intersection for two lines is the solution for the given system
5. Quadratic equation or equation of second degree has the form $ax + bx + c = 0$ $a, b, c \in \mathbb{R}$(a≠0)where the sum of two roots is $s = -\frac{b}{a}$ and their product is $p = \frac{c}{a}$ To solve this equation, first we find the discriminant (delta):

   There are three cases:$\Delta = b^2 - 4ac$

- If $\Delta>0$ there are two distinct real roots:

  $x_1=\frac{-b+\sqrt{\Delta}}{2a}$ and $x_2=\frac{-b-\sqrt{\Delta}}{2a}$

- If $\Delta=0$, there is one repeated real root (one double root):

  $x_2=x_2\frac{-b}{2a}$

- If $\Delta<0$, there is no real root.

6. If the two real roots exist, i.e $\Delta\geq 0$, then the factor form is $ax_2+bx+c=a(x-x_1)(x+x_2)$.

   If there is no real root, i.e $\Delta<0$, then the expression ax^2-bx-c has no factor form.

7. Biquadratic equation $ax^{2n}+x^n+c=0$ is solved by letting $y=x^2$

8. Nested radical $\sqrt{A\pm\sqrt{B}}$ is denested by letting $\sqrt{A\pm\sqrt{B}}=\sqrt{x}\pm\sqrt{y}$

9. Irrational equation is the equation involving radical sign. We solve irrational equations by squaring both sides. By substituting all obtained solutions in the given equation, those which dont satify the given equation are rejected.

10. Application

    Supply and demand analysis

    Market equilibrium is when the amount of product produced is equal to the amount of quantity demanded. Max can then figure out how to price his new lemonade products based on market equilibrium.

    Linear motion

    Linear motion is a motion along a straight line, and can therefore be described mathematically using only one spatial dimension.

Balancing equation

In chemistry, to balance the chemical equation we set the reactants and products equal to each other.

Calculating Areas

People frequently need to calculate the area of things like rooms, boxes or plots of land.

Figuring Out a Profit

Sometimes calculating a business' profit requires using a quadratic function. If you want to sell something (even something as simple as lemonade) you need to decide how many things to produce so that you'll make a profit.

Quadratics in Athletics

In athletic events that involve throwing things, quadratic equations are highly useful.

Finding a Speed

Quadratic equations are also useful in calculating speeds. Avid kayakers, for example, use quadratic equations to estimate their speed when going up and down a river.

Revision exercise

1. Solve the following equations in set of real numbers;

   a) $3x+82=6-x$ b) $4x(x+9)=0$ c) $11x-18=9+8x$

   d) $\frac{x+3}{2x-1}=4$ e) $24=15+\frac{x}{10}$ f) $\frac{x+2}{3}=4x-3$

   g) $5x-6=22-2x$ h) $6x+9=33-2x$

2. Solve the following inequalities.

   a) $x+4<3$ b) $2x+6>8$ c) $\frac{x+2}{x}<0$

   d) $3x/\ 6\leq 1$ e) $\frac{4x-4}{x+1}\geq 0$ f) $(x+1)(-x-4)<0$

   g) $(-x-9)(x+1)(2x-4)>0$

3. Solve the following equations and inequalities in set of real numbers

a) $x^2-17x+70=0$ b) $4x^2+45x=34$ c) $x^2-10x+1=0$

d) $x^2-7x+10\geq 0$ e) $6x^2-5x+1\leq 0$ f) $x^2+2x+1\geq 0$

g) $x^2+2x<-1$ h) $2x^2+3x+10<-10$ i) $2x^2+3x+20>0$

j) $\frac{x^2-5x+6}{x+1}\leq 0$ k) $\frac{x^2-5x+6}{x^2+1}>0$

4. Solve in set of real numbers the following equations

a) $\sqrt{2x+4}=8$ b) $4+\sqrt{3x-1}=9$

c) $\sqrt{x-1}=\sqrt{2x+1}$ d) $\sqrt{2x-1}=x$

e) $\sqrt{3-x}=x-3$ f) $\sqrt{4x+1}=x+1$

g) $\sqrt{x^2+3}=x+4$

5. Solve in set of real numbers the following equations

a) $2x^4+5x^2-3=0$ b) $x-4x^{\frac{1}{2}}-5=0$

c) $2x+3x^{\frac{1}{2}}+1=0$ d) $2x^{\frac{2}{3}}-9x^{\frac{1}{3}}-5=0$

6. Denest (simplify) the following nested radicals

a) $\sqrt{\left(1-\sqrt{2}\right)^2}$ b) $\sqrt{7+2\sqrt{12}}$ c) $\sqrt{15-2\sqrt{56}}$

d) $\sqrt{12+2\sqrt{32}}$ e) $\sqrt{9-\sqrt{72}}$

7. The senior classes at High School A and High School B planned separate trips to Akagera National Park. The senior class at High School A rented and filled 1 van and 6 buses with 372 students. High School B rented and filled 4 vans and 12 buses with 780 students. Each van and each bus carried the same number of students. How many students can a van carry? How many students can a bus carry?

8. Brenda's school is selling tickets to a spring musical. On the first day of ticket sales the school sold 3 senior citizen tickets and 9 child tickets for a total of \$75. The school took in \$67 on the second day by selling 8 senior citizen tickets and 5 child tickets. What is the price each of one senior citizen ticket and one child ticket?
9. A number is divided into two parts, such that one part is 10 more than the other. If the two parts are in the ratio 5 : 3, find the number and the two parts.
10. Robert's father is 4 times as old as Robert. After 5 years, father will be three times as old as Robert. Find their present ages.
11. The sum of two consecutive multiples of 5 is 55. Find these multiples.
12. The difference in the measures of two complementary angles is 12°. Find the measure of the angles.
13. The cost of two tables and three chairs is \$705. If the table costs \$40 more than the chair, find the cost of the table and the chair.
14. The velocity v m/s of a ball thrown directly up in the air is given by v = 20 – 5t, where t is the time in seconds. At what times will the velocity be between 5 m/s and 15 m/s?
15. A rectangular room fits at least 7 tables that each have 1 square meter of surface area. The perimeter of the room is 16 m. What could the width and length of the room be?
16. A picture has a height that is $\frac{4}{3}$ of its width. It is to be enlarged to have an area of 192 square metres. What will be the dimensions of the enlargement?
17. The product of two consecutive negative integers is 1122. What are the numbers?
18. A garden measuring 12 meters by 16 meters is to have a pedestrian pathway installed all around it, increasing the total area to 285 square meters. What will be the width of the pathway?
19. You have to make a square-bottomed, unlidded box with a height of three metres and a volume of approximately 42 cubic metres. You will be taking a piece of cardboard, cutting three- metres squares from each corner, scoring between the corners, and folding up the edges. What should be the dimensions of the cardboard, to the nearest quarter metres?

Unit 4

Polynomial, Rational and irrational functions

My goals

By the end of this unit, I will explain:

- Generalities on numerical functions.
- Application of rational and irrational functions

Introduction

A polynomial is an expression that can have constants, variables and exponents, that can be combined using addition, subtraction, multiplication and division, but:

- no division by a variable.
- a variable's exponents can only be 0,1,2,3,... etc.
- it can't have an infinite number of terms.

Polynomials are used to describe curves of various types; people use them in the real world to graph curves.

For example, roller coaster designers may use polynomials to describe the curves in their rides. Polynomials can be used to figure how much of a garden's surface area can be covered with a certain amount of soil. The same method applies to many flat-surface projects, including driveway, sidewalk and patio construction. Polynomials can also be used to model different situations, like in the stock market to see how prices will vary over time. Business people also use polynomials to model markets, as in to see how raising the price of a good will affect its sales.

Functions are important in calculating medicine, building structures (houses, businesses,…), vehicle design, designing games, to build computers (formulas that are used to plug to computer programs), knowing how much change you should receive when making a purchase, driving (amount of gas needed for travel).

Required outcomes

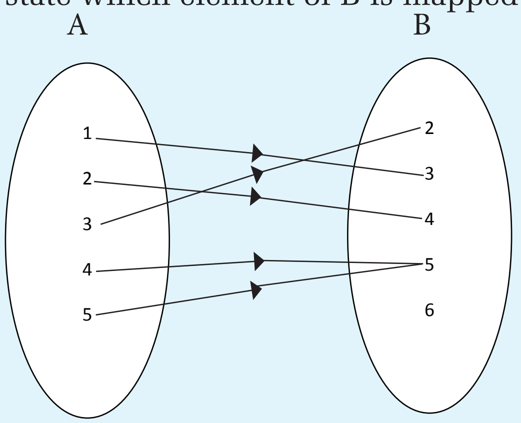
After completing this unit, the learners should be able to:

- Demonstrate an understanding of operations on polynomials, rational and irrational functions, and find the composite of two functions.
- Identify a function as a rule and recognize rules that are not functions.
- Determine the domain and range of a function
- Find whether a function is even , odd , or neither
- Construct composition of functions.

1. Generalities on numerical functions

Activity 1

In the following arrow diagram, for each of the elements of set A, state which element of B is mapped to it.



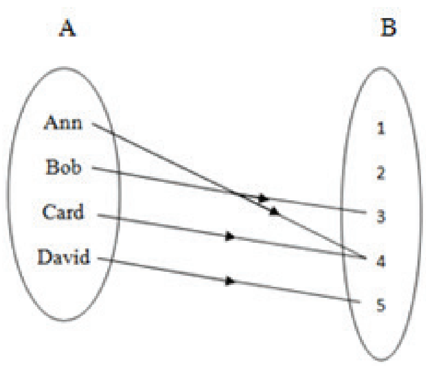
A function is a rule that assigns to each element in a set A one and only one element in set B) We can even define a function as any relationship which takes one element of one set and assigns to it one and only one element of second set. The second set is called a **co-domain**. The set A is called the **domain**, denoted by ***Domf.***

If x is an element in the domain of a function f, then the element that f associates with x is denoted by the symbol $f(x)$ (**read f of x**) and is called the **image of x under f** or the **value of f at x**.

The set of all possible values of $f(x)$ as x varies over the domain is called the **range** of f and it is denoted $R(f)$.

Example 1

Four children, Ann, Bob, Card and David, are given a spelling test which is market out of 5; their marks for the test are shown in the arrow diagram:

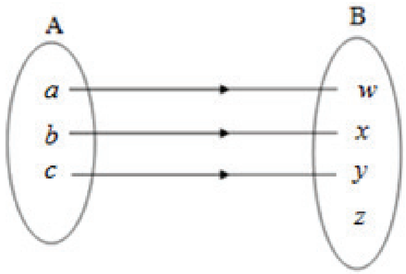


A function f can associate more than one element of the domain onto the same element of the range. For example, $\text{Ann} \to 4$ and $\text{Card} \to 4$. Such functions are said to be **many-to-one.**

Functions for which each element of the domain is associated onto a different element of the range are said to be **one-to-one.** Relationships which are **one-to-many** can occur, but from our preceding definition, they are **not functions.**

Example 2

One-to-one function.

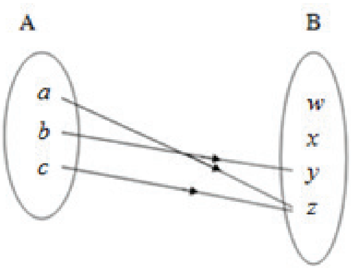


The domain is $\{a,b,c\}$

The codomain is $\{w,x,y,z\}$

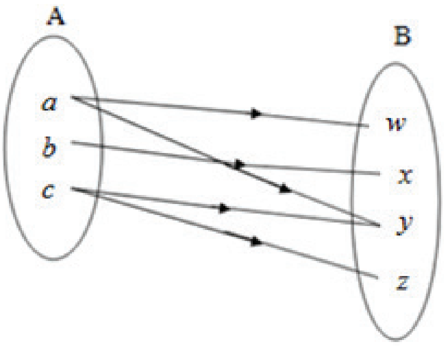
The range is $\{w,x,y\}$

Many-to-one function



The domain is $\{a,b,c\}$, the codomain is $\{w,x,y,z\}$ and the range is $\{y,z\}$

One-to-many relationship



Because this is a one-to-many relationship, it is not a function.

We shall write $f(x)$ to represent the image of x under the function *f*. The letters commonly used for this purpose are *f*, *g* and *h*.

Example 3

Given that $f(x)=x^2$,

find the values of

$f(0), f(2), f(3), f(4)$ *and* $f(5)$

Solution

$f(0)=0^2=0 \quad f(2)=2^2=4$

$f(3)=3^2=9 \quad f(4)=4^2=16$

$f(5)=5^2=25$

Note:

$f(x)=x^2$ can also be written as $f:x\rightarrow x^2$ which is read as "f is a function which maps x onto x^2"

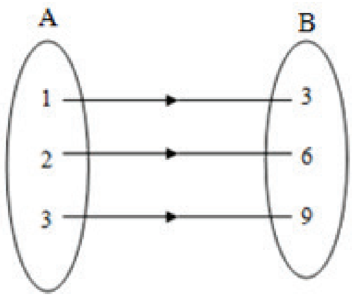
Example 4

Draw arrow diagrams for the functions. Use the domain $\{1,2,3\}$;

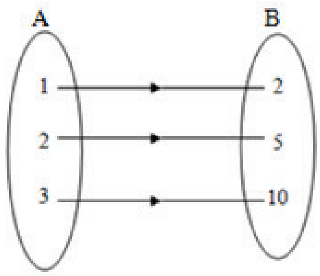
a) $f:x\rightarrow 3x$ b) $h:x\rightarrow x^2+1$ c) $g:x\rightarrow 2x+1$

Solution

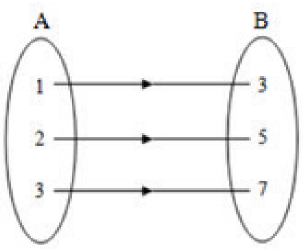
a) $f:x\rightarrow 3x$



b) $h:x\rightarrow x^2+1$



c) $g:x\rightarrow 2x+1$



Example 5

The functions f and g are given as

$f(x) = x + 3 \;\; for \;\; x \geq 0$

and

$g(x) = x^2 \;\; for \;\; -2 \leq x \leq 3$

State the range of each of these functions.

Solution

If $x \geq 0$, then $x + 3 \geq 3$.

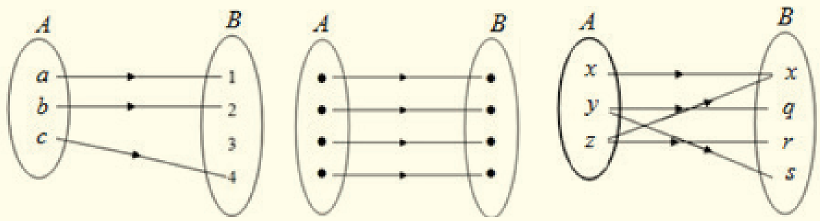
Thus, the range of f will be $f(x) \geq 3$

If $-2 \leq x \leq 3$, then $0 \leq x^2 \leq 9$

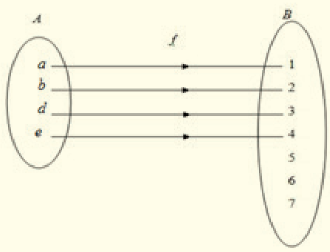
Thus, the range of g will be $0 \leq g(x) \leq 9$

Exercise 1

1. State which of the following relations shows a function.



2. In the following arrow diagram, state the domain, co-domain and range



3. If $f(x) = 2x + 4$, find;

   a) $f(2)$ b) $f(-2)$

   c) $f(d)$ d) The value of a if $f(a) = a$

4. You have ever followed a speech talking about **NDI UMUNYARWANDA** . You have been said that Rwandans have been divided and now they want to be unified. From the types of relationship (in Mathematics), complete this sentence: We have been made……to……by colonialists, **NDI UMUNYARWANDA** is making us……to……

Classification of functions

Activity 2

State which of the following functions is a polynomial, rational or irrational function

1. $f(x)=(x+1)^2$ 2. $h(x)=\dfrac{x^3+2x+1}{x-4}$ 3. $f(x)=\sqrt{x^2+x-2}$

a) Constant function

A function that assigns the same value to every member of its domain is called **a constant function C**.

Example 6

The function f given by $f(x)=3$ is constant.

Remark:

The constant function that assigns the value c to each real number is sometimes called **the constant function c**.

Example 7

The function $f(x)=5$ is called **constant function** 5.

b) Monomial

A function of the form cx^n, where c is constant and n a nonnegative integer is called a **monomial in x**.

Example 8

$2x^3$; πx^7; $4x^0$; $-6x$ and x^{17} are monomials

The functions $4x^{\frac{1}{2}}$ and x^{-3} are not monomials because the powers of x are not nonnegative integers.

c) Polynomial

A function that is expressible as the sum of finitely many monomials in x is called **polynomial in *x***.

Example 9

x^3+4x+7 ; $17-\frac{2}{3}x$; y and x^5 are polynomials. Also $(x-2)^3$ is a polynomial in x because it is expressible as a sum of monomials.

In general, f is a polynomial in x if it is expressible in the form $f(x)=a_0+a_1x+a_2x^2+......+a_nx^n$ where n is a nonnegative integer and $a_0, a_1, ..., a_n$ are real constants.

A polynomial is called

- **linear** if it has the form $a_0+a_1x,\ a_1\neq 0$
- **quadratic** if it has the form $a_0+a_1x+a_2x^2,\ a_2\neq 0$
- **cubic** if it has the form $a_0+a_1x+a_2x^2+a_3x^3,\ a_3\neq 0$

d) Rational function

A function that is expressible as a ratio of two polynomials is called **rational function**. It has the form $\frac{a_0+a_1x+a_2x^2+......+a_nx^n}{b_0+b_1x+b_2x^2+......+b_mx^m}$.

Example 10

$$f(x)=\frac{x^2+4}{x-1},\ g(x)=\frac{1}{3x-5}$$ are rational functions

e) Irrational function

A function that is expressed as root extractions is called irrational function. It has the form $\sqrt[n]{f(x)}$, where $f(x)$ is a polynomial and n is positive integer greater or equal to 2.

Example 11

$f(x)=\dfrac{\sqrt{x^2+4}}{\sqrt[3]{x-1}}$, $g(x)=\sqrt{\dfrac{x}{x-5}}$ are irrational functions

Finding domain of definition

Activity 3

For which value(s) the following functions are not defined:

1. $f(x)=x^3+2x+1$
2. $f(x)=\dfrac{1}{x}$
3. $g(x)=\dfrac{x+2}{x-1}$

Case 1: The given function is a polynomial

Given that $f(x)$ is polynomial, then the domain of definition is the set of real numbers. That is $Domf=\mathbb{R}$

Case 2: The given function is a rational function

Given that $f(x)=\dfrac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are polynomials, then the domain of definition is the set of real numbers excluding all values where the denominator is zero. That is $Domf=\{x\in\mathbb{R}: h(x)\neq 0\}$

Exercise 2

Find the domain of definition for each of the following functions:

1. $f(x)=x^3+2x^2-2$
2. $g(x)=-2$
3. $h(x)=\dfrac{x^3+2x^2-2}{x-5}$
4. $f(x)=\dfrac{x^2-2}{x^2-8x+15}$
5. $f(x)=(x+6)^2$

Case 3: The given function is an irrational function

Activity 4

For each of the following functions, give a range of values of the variable x for which the function is not defined.

1. $f(x)=\sqrt{2x+1}$
2. $f(x)=\sqrt[3]{x^2+x-2}$
3. $g(x)=\sqrt{\dfrac{x-2}{x+1}}$

Given that $f(x)=\sqrt[n]{g(x)}$ where $g(x)$ is a polynomial, there are two cases:

a) If n is odd number, then the domain is the set of real numbers. That is $Domf = \mathbb{R}$

b) If n is even number, then the domain is the set of all values of x such that $g(x)$ is positive or zero. That is $Domf = \{x \in \mathbb{R}: g(x) \geq 0\}$

Example 12

The domain of the function $f(x)=3x^5+2x^4+4x+6$ is $\mathbb{R}$ since it is a polynomial.

Example 13

Given $f(x)=\dfrac{x+1}{3x+6}$, find the domain of definition.

Solution

Condition: $3x+6 \neq 0$

$3x+6=0 \Rightarrow x=-2$

Then, $Domf = \mathbb{R} \setminus \{-2\}$ or $Domf =]-\infty-2[\cup]-2,+\infty[$

Example 14

Given $f(x)=\sqrt{x^2-1}$, find domain of definition.

Solution

Condition: $x^2-1\geq 0$.

We need to construct a sign table to see where x^2-1 is positive

$x^2-1=0 \Rightarrow x=\pm 1$

x	$-\infty$		-1		1		$+\infty$
x^2-1		+	0	-	0	+	

Thus, $Domf=]-\infty,-1]\cup[1,+\infty[$

Example 15

Find domain of definition of $f(x)=\sqrt[3]{x+1}$

Solution

Since the index in radical sign is odd number, then

$Domf=\mathbb{R}$

Example 16

What is the domain of definition of $g(x)$ if $g(x)=\sqrt[4]{x^2+1}$?

Solution

Condition: $x^2+1\geq 0$

Clearly x^2+1 is always positive.

Thus $Domg=\mathbb{R}$

Example 17

Find domain of $f(x)=\dfrac{x}{\sqrt{x^3-4x^2+x+6}}$

Solution

Condition: $x^3-4x^2+x+6>0$.

Here we have combined two conditions: $x^3-4x^2+x+6\geq 0$ and $x^3-4x^2+x+6\neq 0$

$x^3-4x^2+x+6=(x+1)(x-2)(x-3)$'

x	$-\infty$		-1		2		3		$+\infty$
$x+1$		-	0	+	+	+	+	+	
$x-2$		-	-	-	0	+	+	+	
$x-3$		-	-	-	-	-	0	+	
x^3-4x^2+x+6		-	0	+	0	-	0	+	

Then, $Domf =]-1,2[\cup]3,+\infty[$

Example 18

The following functions map an element ***x*** of the domain onto its image y. i.e $f: x \to y$

For each of the three functions below, state

(i) the domain for which the function is defined,

(ii) the corresponding range of the function,

(iii) whether the function is one-to-one or many-to-one.

a) $f: x \to x+3$ b) $f: x \to \sqrt{x}$ c) $f: x \to \frac{1}{x^2}$

d) $f: x \to x+3$

Solution

a) $x \to x + 3$

(i) The function is defined for all real numbers, so the domain is $\mathbb{R}$.

(ii) For this domain, the range will contain all real numbers, so the range is $\mathbb{R}$.

(iii) Each element of the range is obtained from only one element of the domain, so the function is **one-to-one (1 to 1)**.

b) $f: x \to \sqrt{x}$

(i) The function is not defined for negative x, so the domain is $\{x \in \mathbb{R} : x \geq 0\}$, $Domf = [0,+\infty[$

(ii) The range will contain all positive numbers in $\mathbb{R}$.

The range is therefore $\{y \in \mathbb{R} : y \geq 0\}$, $R(f) = [0,+\infty[$

(iii) Each element of the range is obtained from only one element of the domain, so the function is **one-to-one (1 to 1)**.

c) $f: x \to \frac{1}{x^2}$

(i) The function is defined for all real x except for $x=0$. We write the domain as $\{x \in \mathbb{R} : x \neq 0\}$ or

$Domf =]-\infty, 0[\cup]0, +\infty[$

(ii) For this domain, the range will contain neither zero nor any negative numbers because x^2 will be positive.

The range is therefore $\{y \in \mathbb{R} : y > 0\}$ or $R(f) =]0, +\infty[$

(iii) Here the element of the domain can be obtained by more than one element of the domain. For example,

$f(2) = f(-2) = \frac{1}{4}$. So the function is **many-to-one.**

Exercise 3

Find the domain of definition for each of the following functions;

1. $f(x) = \sqrt{4x-8}$
2. $g(x) = \sqrt{x^2+5x-6}$
3. $h(x) = \frac{x^3+2x^2-2}{\sqrt[3]{x+4}}$
4. $f(x) = \frac{x-2}{\sqrt[4]{x^2-25}}$
5. $f(x) = \sqrt{\frac{(x-1)^2}{x+4}}$

Operations on functions

Activity 5

Given the functions $f(x) = \frac{x+1}{2x-3}$ and $g(x) = x+1$, find;

1. $f(x)+g(x)$
2. $f(x)-g(x)$
3. $f(x)\cdot g(x)$
4. $\frac{f(x)}{g(x)}$

Just as numbers can be added, subtracted, multiplied and divided to produce other numbers, there is a useful way of adding, subtracting, multiplying and dividing functions to produce other functions. These operations are defined as follows:

Given functions f and g, **sum** $f+g$, **difference** $f-g$, **product** $f\cdot g$ and **quotient** $\frac{f}{g}$, are defined by

- $(f+g)(x)=f(x)+g(x)$
- $(f-g)(x)=f(x)-g(x)$
- $(f\cdot g)(x)=f(x)\cdot g(x)$
- $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$

For the functions, $f+g$, $f-g$ and $f\cdot g$, the domain is defined to be the intersection of the domains of f and g and for $\frac{f}{g}$, as we have seen it, the domain is this intersection with the points where $g(x)=0$ excluded.

Example 19

Let f and g be the functions $f(x)=3x^4-5x^3+x-4$ and $g(x)=4x^3-3x^2+4x+3$. Find $(f+g)(x)$ and $(f-g)(x)$

Solution

$$
\begin{array}{rl}
+ & f(x)=3x^4-5x^3 \qquad +x\ -4 \\
 & g(x)= \qquad 4x^3-3x^2+4x+3 \\
\hline
 & (f+g)(x)=3x^4-x^3-3x^2+5x-1
\end{array}
\qquad
\begin{array}{rl}
- & f(x)=3x^4-5x^3 \qquad +x\ -4 \\
 & g(x)= \qquad 4x^3-3x^2+4x+3 \\
\hline
 & (f-g)(x)=3x^4-9x^3+3x^2-3x-7
\end{array}
$$

Example 20

If $f(x)=\frac{9}{x+2}$ and $g(x)=x^3$. Find;

a) $h(x)=f(x)+g(x)$

b) $t(x)=f(x)\cdot g(x)$

c) $k(x)=\frac{f(x)}{g(x)}$

Solution

a. $h(x)=f(x)+g(x)$

$=\dfrac{9}{x+2}+x^3$

$=\dfrac{9+x^3(x+2)}{x+2}$

$=\dfrac{x^4+2x^3+9}{x+2}$

b. $t(x)=f(x)\cdot g(x)$

$=\dfrac{9}{x+2}\cdot x^3$

$=\dfrac{9x^3}{x+2}$

c. $k(x)=\dfrac{f(x)}{g(x)}$

$=\dfrac{\frac{9}{x+2}}{x^3}$

$=\dfrac{9}{x+2}\times\dfrac{1}{x^3}$

$=\dfrac{9}{x^4+2x^3}$

Example 21

Let f and g be the functions $f(x)=\sqrt{5-x}$ and $g(x)=\sqrt{x-3}$

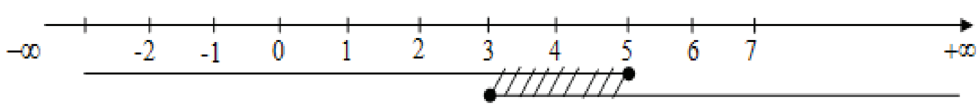
Then the formulae for $f+g$, $f-g$, $f\cdot g$ and $\dfrac{f}{g}$ are;

- $(f+g)(x)=f(x)+g(x)=\sqrt{5-x}+\sqrt{x-3}$
- $(f-g)(x)=f(x)-g(x)=\sqrt{5-x}-\sqrt{x-3}$
- $(f\cdot g)(x)=f(x)\cdot g(x)=\sqrt{5-x}\cdot\sqrt{x-3}$
- $\left(\dfrac{f}{g}\right)(x)=\dfrac{f(x)}{g(x)}=\dfrac{\sqrt{5-x}}{\sqrt{x-3}}$

Since the domain of f is $(-\infty,5]$ and the domain of g is $[3,+\infty)$ the domain of $f\pm g$ and $f\cdot g$ is $[3,5]$. Because this is the intersection of $(-\infty,5]$ and $[3,+\infty)$.

Since $g(x)=0$ when $x=3$, we must exclude this point to obtain the domain of $\dfrac{f}{g}$ which is $(3,5]$.

It will be easy to find this intersection if we use the number line:



The intersection is $[3,5]$but remember that $x\neq 3$, so domain is $]3,5]$.

Example 22

Let $f(x)=3\sqrt{x}$ and $g(x)=\sqrt{x}$, find $(f\cdot g)(x)$

Solution

The formula for $f\cdot g$ is

$$(f\cdot g)(x)=f(x)\cdot g(x)$$
$$=\sqrt{x}\sqrt{x}$$
$$=x$$

$(f\cdot g)(x)=3x$ makes sense and yields real numbers for all x, the formula alone will not correctly describe the domain of $f\cdot g$ for us, we must write $(f\cdot g)(x)=3x, x\geq 0$. Because the domain of f is $[0,+\infty)$ and the domain of g is $[0,+\infty)$, the domain of $f\cdot g$ is also $[0,+\infty)$ since this is the intersection of the domains of f and g.

Example 23

Find domain of $f(x)=\sqrt{\frac{x-}{x+3}}-\frac{x}{2x+4}\sqrt{x+}$

Solution

To find the domain of this function, we need to divide it into other functions.

Let $g(x)=\sqrt[3]{\frac{x-2}{x+3}}$, $h(x)=\frac{x}{2x+4}$, *and* $k(x)=\sqrt{x+5}$

such that $f(x)=g(x)-h(x)k(x)$

For $g(x)=\sqrt[3]{\frac{x-2}{x+3}}$:

$x+3\neq 0 \Rightarrow x\neq -3$, $Dom g=\mathbb{R}\setminus\{-3\}$ or

$Domg =]-\infty, -3[\cup]-3, +\infty[$

[*Index in radical sign is odd number*]

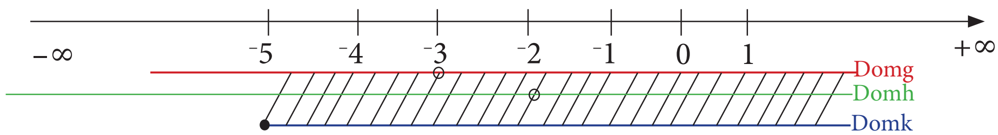
For $h(x) = \dfrac{x}{2x+4}$:

$2x+4 \neq 0 \Rightarrow x \neq -2$, $Domh =]-\infty, -2[\cup]-2, +\infty[$

For $k(x) = \sqrt{x+5}$:

$x+5 \geq 0 \Rightarrow x \geq -5$, $Domk = [-5, +\infty[$

Now, $Domf = Domg \cap Domh \cap Domk$



The intersection is $[-5, +\infty[\setminus \{-3, -2\}$.

Thus, $Domf = [-5, +\infty[\setminus \{-3, -2\}$ or

$Domf = [-5, -3[\cup]-3, -2[\cup]-2, +\infty[$

Exercise 4

1. Given the functions $f(x) = 2x^3 + 5x - 1$ and $g(x) = 3x - 4$, find $(f+g)(x)$
2. Given the functions $f(x) = 3x^3 - 5x^2 + 7x - 4$ and $g(x) = 2x^2 - x + 3$, find $(f \cdot g)(x)$
3. Find the domain of definition of $(f+g)(x)$ if $f(x) = \sqrt{2x+3}$ and $g(x) = \dfrac{x^2 - x + 1}{3x+9}$
4. Find the domain of definition of $\left(f + \dfrac{g}{h}\right)(x)$ if $f(x) = \sqrt[3]{x^3 + 2x + 1}$ $g(x) = x - 7$ and $h(x) = \sqrt{2x+8}$
5. Given the functions $f(x) = 2x^3 + 5x - 1$, $g(x) = 4x^2 - 13x - 8$ and $h(x) = x - 3$, find $(f + g - h)(x)$

Further considerations

Odd and even functions

Activity 6

For each of the following functions, find $f(-x)$ and $-f(x)$. Compare $f(-x)$ and $-f(x)$ using $=$ or $\neq$;

1. $f(x)=x^2+2x+3$ 2. $f(x)=\sqrt[3]{x^3+x}$ 3. $f(x)=\frac{x^2-3}{x^2+1}$

Even Function

A function $f(x)$ is said to be **even** if the following conditions are satisfied

- $\forall x \in Dom f, -x \in Dom f$
- $f(-x)=f(x)$

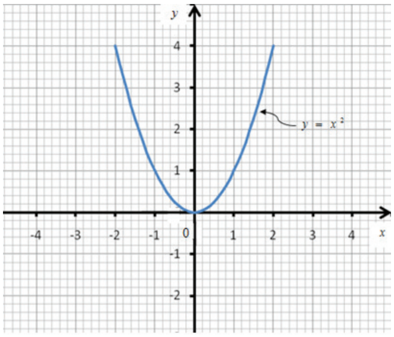
The graph of such function is **symmetric about the vertical axis.** i.e $x=0$

Example 24

The function $f(x)=x^2$ is an even function since $\forall x \in Dom f=\mathbb{R}, -x \in Dom f=\mathbb{R}$

and

$f(-x)=(-x)^2=x^2=f(x)$



Odd function

A function $f(x)$ is said to be **odd** if the following conditions are satisfied

- $\forall x \in Dom f, -x \in Dom f$
- $f(-x)=-f(x)$

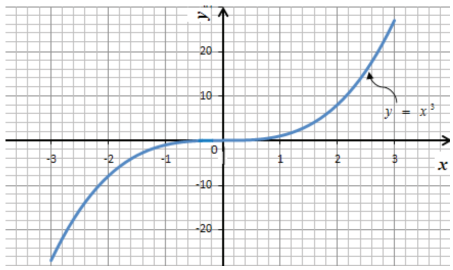
The graph of such a function looks the same when rotated through half a revolution about 0. This is called **rotational symmetry**.

Example 25

$f(x) = x^3$ is an odd function since $\forall x \in Domf = \mathbb{R}, -x \in Domf = \mathbb{R}$

and

$f(-x) = (-x)^3 = -x^3 = -f(x)$



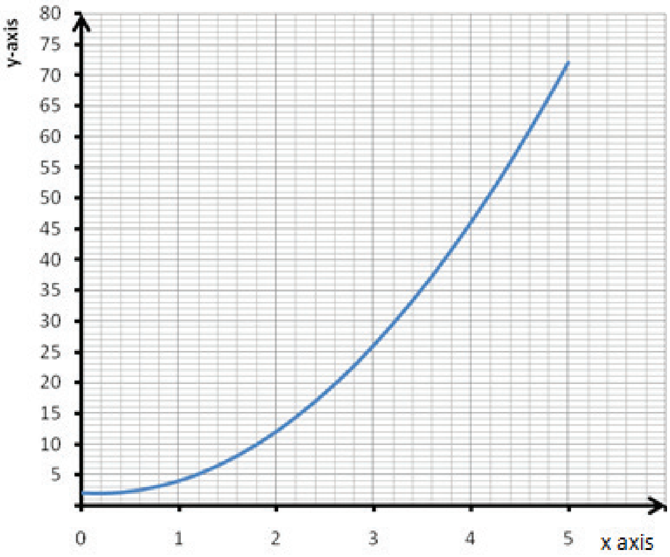
Example 26

Consider the function $f(x) = 3x^2 - |x| + 2$ for $x \geq 0$. Is this function odd, even or neither?

Here, the domain of the given function is restricted to $Domf = [0, +\infty[$ since $x \geq 0$. $\forall x \in Domf, -x \notin Domf$.

Thus, the given function is neither even nor odd.

Here is the graph



Exercise 5

Study the parity of the following functions:

1. $f(x)=2x^2+2x-3$
2. $f(x)=\dfrac{3x^3+2x^2+8}{x-5}$
3. $g(x)=x^3-x$
4. $h(x)=\dfrac{x^2+4}{x^2-4}$
5. $g(x)=x(x^2+x)$

Composite functions

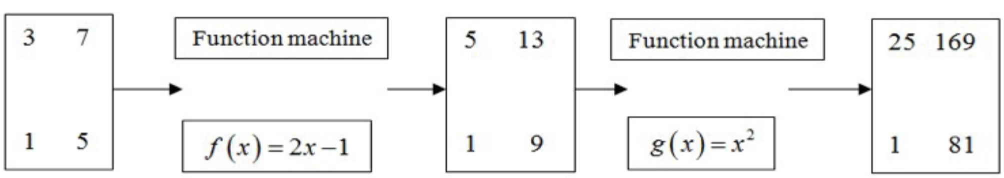
Activity 7

Consider two functions $f(x)=3x+2$ and $g(x)=x^3-1$ Find

1. $f[g(x)]$
2. $g[f(x)]$
3. Compare the two results

Consider the functions $f(x)=2x-1$ and $g(x)=x^2$

Using the **machine**, suppose that we input the numbers $\{1,3,5,7\}$ into machine f and then take the output from machine f and put it into machine g.



This **combined** or **composite** function is written $(g \circ f)(x)$ or $g[f(x)]$or simply gf. The function f is performed first and so is written nearer to the variable $\boldsymbol{x}$. The set $\{1,3,5,7\}$ is the domain for the composite function and $\{1,25,81,168\}$ is the range.

Note that $(f \circ g)(x) \neq (g \circ f)(x)$

Example 27

If $f(x)=2x$ and $g(x)=3x+1$, express $\boldsymbol{gof}$ as a single function $h(x)$.

Solution

$f(x)=2x$ so $(g\circ f)(x)=g(2x)=3(2x)+1=6x+1$
$\therefore h(x)=6x+1 \therefore h(x)=6x+1$

Example 28

Let $f(x)=x-1$ *and* $g(x)=\sqrt{x}$ *, find* $(fog)(x)$ *and* $(gof)(x)$

- $g(x)=\sqrt{x}$, *so* $fg(x)=f(\sqrt{x})=\sqrt{x}-1$ $\therefore (g\,o\,f)(x)=\sqrt{x-1}$
- $f(x)=x-1$, *so* $gf(x)=g(x-1)=\sqrt{x-1}$ $\therefore (f\,o\,g)(x)=\sqrt{x}-1$

Example 29

If $f(x)=3x$ and $g(x)=x^2+1$, find

$(f\circ g)(x)$

$(g\circ f)(x)$

$(g\circ f)(4)$

Solution

a. $(f\circ g)(x)=f[g(x)]=3(x^2+1)=3x^2+3$

b. $(g\circ f)(x)=g[f(x)]=(3x)^2+1=9x^2+1$

c. Since $(g\circ f)(x)=9x^2+1$, $(g\circ f)(4)=9(4)^2+1=9(16)+1=145$

Or first we find $f(4)=3(4)=12$ and $g(12)=(12)^2+1=145$, then

$(g\circ f)(4)=145$.

Example 30

If $f(x)=4$ and $g(x)=x+1$. find

a) $(f\circ g)(x)$

b) $(g\circ f)(x)$

Solution

a) $(f\circ g)(x)=f[g(x)]=4$

b) $(g\circ f)(x)=g[f(x)]=4+1=5$

Exercise 6

Find $(f \circ g)(x)$ and $(g \circ f)(x)$

1. If $f(x) = x^3 - 3x^2 + 1$ and $g(x) = 2$
2. If $f(x) = 2x^2 + x - 3$ and $g(x) = 6x$
3. If $f(x) = x^3 + 2x^2 + x - 4$ and $g(x) = 7x + 3$

The inverse of a function

Activity 8

Find the value of x in function of y if

1. $y = x + 1$ 2. $y = 3x - 2$ 3. $y = \dfrac{-x+3}{2x-1}$

Consider a function *f* which maps each element *x* of the domain x onto its image *y* in the range *y* that is $f: x \to y$ where $x \in X, y \in Y$
If this map can be reversed, i.e $f^{-1}: y \to x$ and resulting relationship is a function, it is called **the inverse of the original function**, and is denoted by f^{-1}
Only one-to-one functions can have an inverse function. To find the inverse of one-to-one functions, we change the subject of a formula.

Example 31

Find the inverse function of $f(x) = 2x + 3$.

Solution

Let us make $\boldsymbol{x}$ the subject of $y = 2x + 3$ as follows:

$y = 2x + 3$

Rearranging $y - 3 = 2x$, then $x = \dfrac{y-3}{2}$

$\therefore f^{-1}(x) = \dfrac{x-3}{2}$

Example 32

Find the inverse of the function $f(x)=3x-1$

Solution

If $f(x)=3x-1$, we require $f^{-1}(y)=x$.

If $y=3x-1$ then $x=\dfrac{y+1}{3}$

So, given y, we can return to x using the expression $\dfrac{y+1}{3}$.

Thus, $f^{-1}(x)=\dfrac{x+1}{3}$

Example 33

Find the inverse of the function $g(x)=\dfrac{3}{x-1}$

Solution

If $g(x)=\dfrac{3}{x-1}$, we require $g^{-1}(y)=x$

If $y=\dfrac{3}{x-1}$ then $\dfrac{1}{y}=\dfrac{x-1}{3}$ or $x=\dfrac{3}{y}+1$

So, given y, we can return to x using the expression $\dfrac{3}{y}+1$.

Thus, $g^{-1}(x)=\dfrac{3}{x}+1$

Another method is to write the separate operations of the function as follow chart. We, then, reverse the flow chart, writing the inverse of each operation.

Example 34

Find the inverse of the function $f(x)=2x+1$

Solution

First write the function as flow chart with input x and output $2x+3$, i.e

$\boxed{x} \rightarrow \boxed{\times 2} \rightarrow \boxed{+3} \rightarrow \boxed{2x+3}$

Now, reverse the flow chart and write the inverse of each operation, i.e $\leftarrow \boxed{:2} \leftarrow \boxed{-3} \leftarrow$

With input x, this will now output the inverse function,

$$\boxed{\frac{x-3}{2}} \leftarrow \boxed{:2} \leftarrow \boxed{-3} \leftarrow \boxed{x}$$

So the inverse function f^{-1} is $x \to \frac{x-3}{2}$.

Example 35

Find the inverse of the function $g(x) = 2 - x$

Solution

$$\boxed{x} \to \boxed{\times(-1)} \to \boxed{+2} \to \boxed{2-x}$$

$$\boxed{\frac{x-2}{-1}} \leftarrow \boxed{:(-1)} \leftarrow \boxed{-2} \leftarrow \boxed{x}$$

$\Updownarrow$

$2-x$

So the inverse function is $2-x$. g is its own inverse.

Example 36

Find the inverse of the function $h(x) = \frac{1}{x} - 3$

Solution

$$\boxed{x} \to \boxed{invert} \to \boxed{-3} \to \boxed{\frac{1}{x}-3}$$

$$\boxed{\frac{1}{x+3}} \leftarrow \boxed{invert} \leftarrow \boxed{+3} \leftarrow \boxed{x}$$

So the inverse function is $\frac{1}{x+3}$

Exercise 7

Find the inverse of the following functions

1. $f(x) = 5x + 2$
2. $g(x) = -7x - 2$
3. $h(x) = \frac{-2x+1}{x-2}$

3. Applications

Activity 9

Give three examples of where you think functions can be used in daily life.

Polynomials are used to describe curves of various types; people use them in the real world to graph curves.
For example, roller coaster designers may use polynomials to describe the curves in their rides. Polynomials can be used to figure how much of a garden's surface area can be covered with a certain amount of soil. The same method applies to many flat-surface projects, including driveway, sidewalk and patio construction. Polynomials can also be used to model different situations, like in the stock market to see how prices will vary over time. Business people also use polynomials to model markets, as in to see how raising the price of a good will affect its sales. For people who work in industries that deal with physical phenomena or modeling situations for the future, polynomials come in handy every day. These include everyone from engineers to businessmen. For the rest of us, they are less apparent but we still probably use them to predict how changing one factor in our lives may affect another--without even realizing it.
Functions are important in calculating medicine, building structures (houses, businesses,…), vehicle design, designing games, to build computers (formulas that are used to plug to computer programs), knowing how much change you should receive when making a purchase, driving (amount of gas needed for travel).

Example 37

In zoology, the irrational function $h(x)=0.4\sqrt[3]{x}$ is an approximation of the height h in metres of a female giraffe using her weight x in kilograms. Find the heights of female giraffe with weights of 500 kg and 545 kg.

Solution

Evaluate $h(x)$ for both weights.

$h(500) = 0.4\sqrt[3]{500} \approx 3.17\,m$

$h(500) = 0.4\sqrt[3]{500} \approx 3.17\,m$

The heights are approximately 3.17 m and 3.27 m

Unit summary

1. A function f is a rule that assigns to each element in a set A one and only one element in set B. The second set is called a co-domain. The set A is called the domain, denoted by Domf. The set of all possible values of *f(x)* as *x* varies over the domain is called the range
2. A function that is expressible as ratio of two polynomials is called rational function. It has the form $\frac{a_0 + a_1x + a_2x^2 + ... + a_nx^n}{b_0 + b_1x + b_2x^2 + ... + b_mx^m}$.
3. A function that is expressed as root extractions is called irrational function. It has the form $\sqrt[n]{f(x)}$, where $f(x)$ is a polynomial and n is positive integer greater or equal to 2.
4. Given that $f(x)$ is polynomial, then the domain of definition is the set of real numbers. That is $Domf = \mathbb{R}$
5. Given that $f(x) = \frac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are polynomials, then the domain of definition is the set of real numbers excluding all values where the denominator is zero.
6. Given that $f(x) = \sqrt[n]{g(x)}$ where $g(x)$ is a polynomial, there are two cases
   - If n is odd number, then the domain is the set of real numbers. That is $Domf = \mathbb{R}$
   - If n is even number, then the domain is the set of all values of x such $g(x)$ is positive or zero.
7. Given functions *f* and g, **sum** $f+g$, **difference** $f-g$, **product** $f \cdot g$ and **quotient** $\frac{f}{g}$, are defined by

- $(f+g)(x)=f(x)+g(x)$
- $(f-g)(x)=f(x)-g(x)$
- $(f\cdot g)(x)=f(x)\cdot g(x)$
- $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$

8. A function $f(x)$ is said to be **even** if the following conditions are satisfied
   - $\forall x\in Domf, -x\in Domf$
   - $f(-x)=f(x)$

   The graph of such function is **symmetric about the vertical axis.**

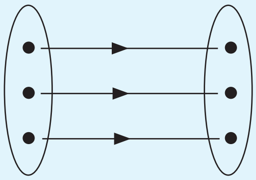
   i.e $x=0$
9. Afunctionissaidtobeoddifthefollowingconditionsaresatisfied
   - $\forall x\in Domf, -x\in Domf$
   - $f(-x)=-f(x)$

   The graph of such a function looks the same when rotated through half a revolution about 0. This is called rotational symmetry.
10. The combined or composite function is written $(gof)(x)$ or $g[f(x)]$or simply gf. The function f is performed first and so is written nearer to the variable x 9.
11. If the map $f:x\to y$ where $x\in X, y\in Y$ can be reversed, i.e $f^{-1}:y\to x$ and resulting relationship is a function, it is called **the inverse of the original function**, and is denoted by f^{-1}. Only one-to-one functions can have an inverse function and to find the inverse of one-to-one functions, we can change the subject of a formula.
12. Polynomials are used to describe curves of various types; people use them in the real world to graph curves. Functions are important in calculating medicine, building structures (houses, businesses,…), vehicle design, designing games, to build computers (formulas that are used to plug to computer programs), knowing how much change you should receive when making a purchase, driving (amount of gas needed for travel).

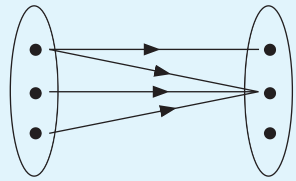
Revision exercise

1. State which of the following arrow diagrams show functions.

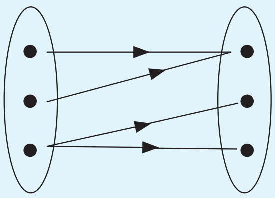
a)



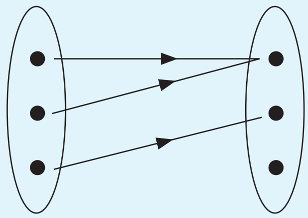
b)



c)

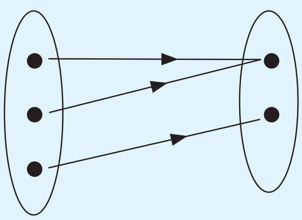


d)

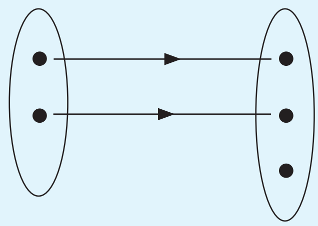


2. State which of the following arrow diagrams show
(i) a one to one function mapping into the co-domain,
(ii) a one to one function mapping onto the co-domain,
(iii) a many to one function mapping into the co-domain.

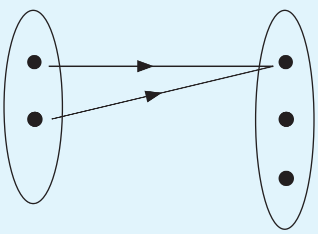
a)



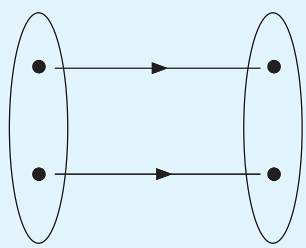
b)



c)



d)



3. Given that $f(x) = 3x^2 + 2$, find:

a) $f(-2)$ b) $f(4)$ c) $f(0)$ d) $f(-\sqrt{3})$ e) $f(t)$

4. Given that $g(x)=\dfrac{x+1}{x-1}$, find:

a) $g(1.1)$ b) $g\left(\dfrac{1}{4}\right)$ c) $g\left(\sqrt[3]{5}+1\right)$ d) $g(\pi)$ e) $g(a-1)$

5. Given that $\phi(x)=\begin{cases}\dfrac{1}{x} & x\geq 3\\ 2x & x<2\end{cases}$, find:

a) $\phi(2)$ b) $\phi(-4)$ c) $\phi(3)$ d) $\phi(3.1)$ e) $\phi(2.9)$

6. If $f(x)=2x+3$, find the value of a if $f(a)=a$

7. If $f(x)=2x^2$ and $g(x)=3-x$, find the possible values of a if $f(a)=g(a)$

8. The function g is given by $g(x)=ax^2-b$. If $g(2)=5$ and $g(-1)=2$, find the values of a and b and hence find $g(-4)$.

9. Each of the following functions map an element x of the domain onto its image y, i.e $f(x)=y$. Find the range of each function for the given domain and state whether the function is one to one or many to one.

a) $f:x\rightarrow x+3$ with domain $\{x:0\leq x\leq 4\}$

b) $f:x\rightarrow x^2$ with domain $\{x:-3\leq x\leq 3\}$

c) $f:x\rightarrow \dfrac{1}{x}$ with domain $\{x:x\geq 1\}$

d) $f:x\rightarrow x^2+4$ with domain $\mathbb{R}$

e) $f:x\rightarrow \dfrac{1}{x-1}$ with domain $\{x\in\mathbb{R}:x\neq 1\}$

10. Find the domain of the given function:

a) $f(x)=\dfrac{1}{x-3}$ b) $g(x)=\dfrac{1}{5x+7}$ c) $g(x)=\sqrt{x^2-3}$

d) $g(x)=\sqrt{(x-1)(x+2)}$ e) $\phi(x)=\sqrt{x^2+3}$ f) $\phi(x)=\dfrac{x}{\sqrt{|x|+1}}$

g) $h(x)=x^2 \quad x<-3$ h) $h(x)=\sqrt{x} \quad x\geq 5$ i) $f(x)=\begin{cases}\sqrt{x} & x\geq 2\\ \dfrac{1}{x-2} & x<2\end{cases}$

j) $f(x)=\begin{cases}\sqrt{-x} & x\leq -3\\ 0 & x\geq 2\end{cases}$

11. Find the domain of the given function:

a) $f(x)=4x^3-3x+1$ b) $g(x)=\dfrac{x+1}{x^2+2x-15}$ c) $h(x)=\sqrt{1-2x}$

d) $k(x)=\sqrt{4-x}+\sqrt{x+3}$ e) $i(x)=\dfrac{\sqrt{3x-1}}{\sqrt{x+3}}$

f) $h(x)=\dfrac{\sqrt{x^3+2x^2-x-2}}{x^2-x}$ g) $f(x)=\dfrac{x^2+1}{x^3-2x-3}\sqrt{\dfrac{x-1}{x^2-2x-8}}$

12. Let $g(x)=x-3$ and let $f(x)=\begin{cases}\dfrac{x^2-9}{x+3} & x\neq -3\\ k & x=-3\end{cases}$

Find k so that $f(x)=g(x)$ for all x.

13. Let $g(x)=x-4$ and $f(x)=\begin{cases}\dfrac{x^2-5x+4}{x-1} & x\neq 1\\ k & x=1\end{cases}$

Find the value of k so that $f(x)=g(x)$ for all x.

14. Given the functions $f(x)=x^3+3x^2-2x-2$ and $g(x)=x^2-x$, find:

a) $(f-g)(x)$ b) $(f+g)(x)$ c) $(f\cdot g)(x)$

15. Study the parity of the following functions:

a) $f(x)=\dfrac{x}{x^2+4}$ b) $f(x)=x^3-1$ c) $g(x)=\sqrt[3]{x^3+3x}$

d) $h(x)=\dfrac{x^3-2x}{x^2-4}$

16. Find $(fog)(x)$ and $(gof)(x)$

a) If $f(x)=x^3$ and $g(x)=6$

b) If $f(x)=x^5-x^4-3$ and $g(x)=4x^2+3$

c) If $f(x)=x^3-2x^2-4$ and $g(x)=x^3+1$

17. Find the inverse of the following functions

a) $f(x)=9x-2$ b) $g(x)=\dfrac{-x-2}{x-5}$ c) $h(x)=\dfrac{x-9}{3x-2}$

Unit 5

Limits of polynomial, rational and irrational functions

My goals

By the end of this unit, I will explain:

- Concepts of limits.
- Indeterminate cases.
- Applications.

Introduction

The limit of a function is a fundamental concept in calculus and analysis concerning the behavior of that function near a particular point.

Suppose that a person is walking over a landscape represented by the graph of the function $f(x)$. His/her horizontal position is measured by the value of x, much like the position given by a map of the land or by a global positioning system. His/her altitude is given by the coordinate y. He/she is walking towards the horizontal position given by $x = \text{p}$. As he/she gets closer and closer to it, he/she notices that his/her altitude approaches L. If asked about the altitude of $x = \text{p}$, he/she would then answer L. It means that her/his altitude gets nearer and nearer to L except for a possible small error in accuracy.

The limit of a function $f(x)$ as x approaches p is a number L with the following property: given any target distance from L, there is a distance from p within which the values of $f(x)$ remain within the target distance.

Limits are also used to find the velocity and acceleration of a moving particle.

Required outcomes

After completing this unit, the learners should be able to:

- Calculate limits of certain elementary functions.
- Apply informal methods to explore the concept of a limit including one sided limits.
- Solve problems involving continuity.
- Use the concepts of limits to determine the asymptotes to the rational and polynomial functions.
- Develop calculus reasoning.

1. Concepts of limits

Neighbourhood of a real number

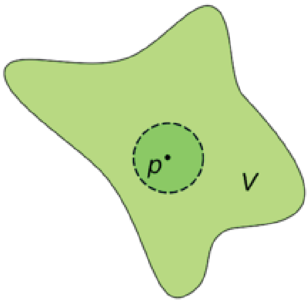
Activity 1

Study the following political map of Lesotho, Swaziland and South Africa. What can you say about the boundaries of Lesotho and Swaziland?

A set N is called a neighbourhood of point p if there exists an open interval I such that $x \in I \subset N$. The collection of all neighbourhoods of a point is called the **neighbourhood system** at the point.

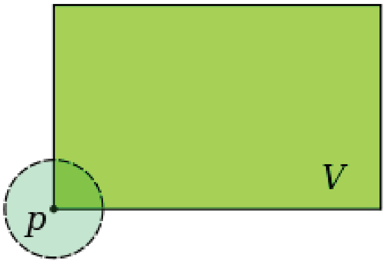
A **deleted neighbourhood** of a point p (sometimes called a **punctured neighbourhood**) is a neighbourhood of p without p itself.

Example 1



A set V in the plane is a neighbourhood of a point p if a small disk around p is contained in V.

Example 2



A rectangle is not a neighbourhood of any of its corner.

Example 3

The interval $(-1,1)=\{y:-1<y<1\}$ is a neighbourhood of $x=0$ in the real line, so the set $(-1,0)\cup(0,1)=(-1,1)\setminus\{0\}$ is a deleted neighbourhood of 0.

Exercise 1

1. Apart from The Kingdom of Lesotho, give two examples of countries or Cities in the world that are surrounded by a single country or city.
2. Give three examples of intervals that are neighbourhoods of -5?
3. Is a circle a neighborhood of each of its points? Why?
4. Draw any plane and show three points on that plane for which the plane is their neighborhood.

Note:

A deleted neighbourhood of a given point is not in fact a neighbourhood of the point. The concept of deleted neighbourhood occurs in the definition of the limit of a function.

Limit of a function

Activity 2

Find:

1. $f(2)$ if $f(x)=\dfrac{x+1}{x+2}$
2. $f(1)$ if $f(x)=\dfrac{\sqrt{x+3}}{\sqrt[3]{3x-2}}$
3. $f(3)$ if $f(x)=4x^3-2x^2+3x-1$

Limits are used to describe how a function behaves as the independent variable moves towards a certain value.

Frequently when studying function $y=f(x)$, we find ourselves interested in the function's behavior near a particular point x_0, but not at x_0.

Example 4

Let us explore numerically how the function $f(x)=\dfrac{x^2-9}{x-3}$ behaves near $x=3$.

Note that $f(x)=\dfrac{x^2-9}{x-3}$ is defined for all real numbers x except for $x=3$. For any $x=3$ we can simplify the expression for $f(x)$ by factoring the numerator and cancelling common factors:

$$f(x)=\frac{(x+3)(x-3)}{x-3}=x+3 \quad \textit{for} \;\; x\neq 3$$

Even though $f(3)$ is not defined, it is clear that we can make the value of $f(x)$ as close as we want to 6 by choosing x close enough to 3. Therefore, we say that $f(x)$ approaches arbitrarily close to 6 as x approaches 3, or, more simply, $f(x)$ approaches the *limit 6* as x approaches 3. We write this as

$$\lim_{x\to 3} f(x)=6 \quad \text{or} \quad \lim_{x\to 3}\frac{x^2-9}{x-3}=6$$

Informally,

If $f(x)$ is defined for all x near a, except possibly at a itself, and if we can ensure the $f(x)$ is as close as we want to L by taking x close enough to a, but not equal to a, we say that the function f approaches the **limit *L*** as x approaches a, and we write

$$\lim_{x\to a} f(x)=L$$

To find limit of a function $f(x)$ as x approaches a, first we need to substitute that value a in the function and see what happens. The limit can exist or not.

Example 5

$$\lim_{x\to 2}(2x+1)=2(2)+1=5$$

Example 6

Since a constant function $f(x)=k$ has the same value k everywhere, it follows that at each point $\lim_{x\to a} k = k$. For example $\lim_{x\to 4} 5 = 5$

Example 7

The limit $\lim_{x\to a} x = a$ is self-evident. For example, $\lim_{x\to -5} x = -5$

$\lim_{x\to 0} x = 0$

Example 8

$$\lim_{x\to 2}\sqrt{x^2-2x+1}=\sqrt{4-4+1}=1$$

Example 9

$$\lim_{x\to 3}\frac{\sqrt{2x+1}}{\sqrt[3]{3x-1}}=\frac{\sqrt{7}}{\sqrt[3]{8}}=\frac{\sqrt{7}}{2}$$

For a rational function $f(x)=\frac{g(x)}{h(x)}$

If x approaches $a\in\mathbb{R}$, we have three cases:

a) $h(x)\neq 0$ for $x=a \Rightarrow \lim_{x\to a} f(x)=f(a)$

b) $g(x)\neq 0$, $h(x)=0$ for $x=a \Rightarrow \lim_{x\to a} f(x)=\infty$

c) $g(x)=0$, $h(x)=0$ for $x=a \Rightarrow \lim_{x\to a} f(x)=\frac{0}{0}$

(Indeterminate form)

Later, we will see how to remove the indeterminate forms.

Example 10

$$\lim_{x\to 2}\frac{x+4}{2+x}=\frac{2+4}{2+2}=\frac{6}{4}=\frac{3}{2}$$

Example 11

$$\lim_{x\to 0}\frac{x^2-2x-3}{x+6}=\frac{0-0-3}{0+6}=-\frac{1}{2}$$

Exercise 2

Evaluate the following limits

1. $\lim_{x\to 1}\left(4x^3-3x^2+2x-1\right)$
2. $\lim_{x\to -4}\frac{x^2-16}{x+4}$
3. $\lim_{x\to 2}\frac{\sqrt{-x+6}}{\sqrt[3]{x^6}}$
4. $\lim_{x\to -1}\frac{3x^2+5}{x}$
5. $\lim_{x\to 2}\frac{x^3+x^2-4x-4}{x-2}$

One sided limits

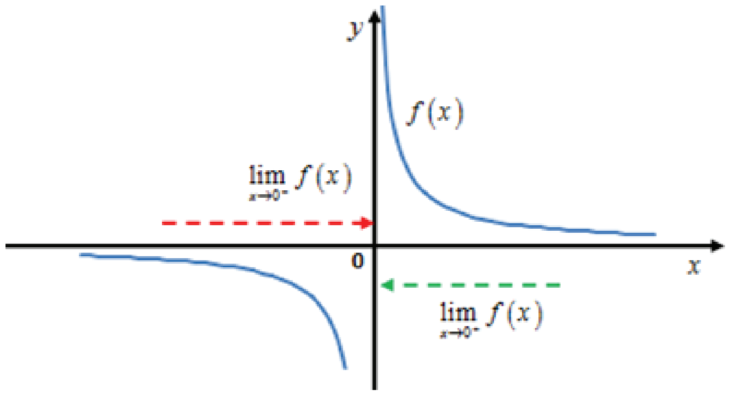
Activity 3

Consider the following function;

$$f(x)=\begin{cases} x-1, & x\le 3 \\ 3x-7, & x>3 \end{cases}$$

Find:

1. $f(2.8)$
2. $f(2.9)$
3. $f(2.99)$
4. $f(3.05)$
5. $f(3.01)$
6. $f(3.001)$



If the value of $f(x)$ approaches L_1 as x approaches x_0 from the right side, we write $\lim\limits_{x\to x_0^+} f(x)=L_1$ and we read "**the limit of $f(x)$ as x approaches x_0 from the right equals L_1**.

If the value of $f(x)$ approaches L_2 as x approaches x_0 from the left side, we write $\lim\limits_{x\to x_0^-} f(x)=L_2$ and we read "**the limit of $f(x)$ as x approaches x_0 from the left equals L_2**.

If the limit from the left side is the same as the limit from the right side, say $\lim\limits_{x\to x_0^-} f(x)=\lim\limits_{x\to x_0^+} f(x)=L$, then we write $\lim\limits_{x\to x_0} f(x)$ and we read "**the limit of $f(x)$ as x approaches x_0 equals L**.

We call $\lim\limits_{x\to x_0^-} f(x)$ *or* $\lim\limits_{x\to x_0^+} f(x)$ **one-side limit**.

Note that $\lim\limits_{x\to x_0} f(x)$ exists if and only if $\lim\limits_{x\to x_0^-} f(x)=\lim\limits_{x\to x_0^+} f(x)$

Example 12

If $f(x)=\frac{|x-2|}{x^2+x-6}$. *Find* $\lim_{x\to 2^-} f(x)$ *and* $\lim_{x\to 2^+} f(x)$

Solution

Observe that $|x-2|=x-2$ *if* $x>2$ and
$|x-2|=-(x-2)$ *if* $x<2$. Therefore,

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}\frac{-(x-2)}{x^2+x-6}=\lim_{x\to 2^-}\frac{-(x-2)}{(x-2)(x+3)}=\lim_{x\to 2^-}\frac{-1}{x+3}=-\frac{1}{5}$$

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}\frac{x-2}{x^2+x-6}=\lim_{x\to 2^+}\frac{x-2}{(x-2)(x+3)}=\lim_{x\to 2^+}\frac{1}{x+3}=\frac{1}{5}$$

Since $\lim_{x\to 2^-} f(x)\neq \lim_{x\to 2^+} f(x)$, then $\lim_{x\to 2} f(x)$ does not exist.

Example 13

Find $\lim_{x\to 3} f(x)$ for $f(x)=\begin{cases} x^2-5 & \text{if } x\le 3 \\ \sqrt{x+13} & \text{if } x>3 \end{cases}$

Solution

As x approaches 3 from the left, the formula for f is

$f(x)=x^2-5$. So that $\lim_{x\to 3^-} f(x)=\lim_{x\to 3}(x^2-5)=4$

As x approaches 3 from the right, the formula for f is

$f(x)=\sqrt{x+13}$ So that;

$$\lim_{x\to 3^+} f(x)=\lim_{x\to 3}\sqrt{x+13}=\sqrt{\lim_{x\to 3}(x+13)}=\sqrt{16}=4$$

Since the one-sided limits are equal, $\lim_{x\to 3} f(x)=4$.

Exercise 3

Evaluate the following limits:

1. $\lim\limits_{x\to 3} f(x)$ if $f(x)=\begin{cases} x^2-2x+1, & x\neq 3 \\ 7, & x=3 \end{cases}$

2. $\lim\limits_{x\to 2} h(x)$ if $h(x)=\begin{cases} x^2-x-1, & x<2 \\ 3x-5, & x\geq 2 \end{cases}$

3. $\lim\limits_{x\to 0} g(x)$ if $g(x)=\begin{cases} x, & x\leq 0 \\ x^2, & x>0 \end{cases}$

4. $\lim\limits_{x\to 1} h(x)$ if $h(x)=\begin{cases} 1, & x>1 \\ 3, & x\leq 1 \end{cases}$

Infinite limits and limits at infinity

Activity 4

1. Consider the following function $f(x)=\dfrac{x+1}{x-1}$. Find:

   a) $f(0.97)$ b) $f(0.98)$ c) $f(0.99)$

   d) $f(1.01)$ e) $f(1.02)$ e) $f(1.03)$

2. Evaluate the following operations:

   a) $-2+\infty$ b) $2-\infty$ c) $-\infty+\infty$

   d) $-\infty\times\infty$ e) $3(-\infty)$ f) $\dfrac{-\infty}{-2}$ g) $\dfrac{\infty}{-\infty}$

Infinite limits

A function whose values grow arbitrarily large can sometimes be said to have an infinite limit. Since infinity is not a number, infinite limits are not really limits at all but they provide a way of describing the behavior of functions that grow arbitrarily large positive or negative.

Example 14

Describe the behavior of the function $f(x)=\frac{1}{x^2}$ near $x=0$.

Solution

As x approaches 0 from either side, the values of $f(x)$ are positive and grow larger and larger, so the limit of $f(x)$ as x approaches 0 does not exist. It is nevertheless convenient to describe the behavior of f near 0 by saying that $f(x)$ approaches ∞ as x approaches zero. We write,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}\frac{1}{x^2}=+\infty \text{ and } \lim_{x\to 0^+} f(x)=\lim_{x\to 0}\frac{1}{x^2}=+\infty$$

Example 15

Describe the behavior of the function $f(x)=\frac{1}{x}$ near $x=0$.

Solution

Let x successively assume values $x=1,\frac{1}{10},\frac{1}{100},\ldots$, then $\frac{1}{x}=1,10,100,\ldots$ successively. As x approaches 0 from the right, the value of $\frac{1}{x}$ gets larger and larger without bound, then $\lim_{x\to 0^+}\frac{1}{x}=+\infty$.

Let x successively assume values $x=-1,-\frac{1}{10},-\frac{1}{100},\ldots$, then $\frac{1}{x}=-1,-10,-100,\ldots$ successively. As x approaches 0 from the left, the value of $\frac{1}{x}$ decreases and becomes more and more negative without bound, then $\lim_{x\to 0^-}\frac{1}{x}=-\infty$

Another way to see this is to construct the sign table:

x	$-\infty$	0	+ $+\infty$
1	+	+	+
$\frac{1}{x}$	-	$\|\|$ ∞	+

Considering the last row, we see that for $x=0$ the value of $\frac{1}{x}$ does not exist (∞). At the left, there is a negative sign, thus $\lim_{x\to 0^-}\frac{1}{x}=-\infty$. At the right there is a positive sign, thus $\lim_{x\to 0^+}\frac{1}{x}=+\infty$.

It follows that $\lim_{x\to 0}\frac{1}{x}$ does not exist because the one sided limits as x approaches zero do not exist.

Example 16

$$\lim_{x\to 4}\frac{2-x}{x^2-2x-8}$$

Solution

As x approaches 4 from the right, the numerator is negative quantity approaching -2 and the denominator, a positive quantity approaching 0. Consequently the ratio is a negative quantity that decreases without bound. That is;

$$\lim_{x\to 4^+}\frac{2-x}{x^2-2x-8}=-\infty$$

As x approaches 4 from the left, the numerator is eventually a negative quantity approaching -2 and the denominator a positive quantity approaching 0. Consequently the ratio is a negative quantity that increases without bound. That is

$$\lim_{x\to 4^-}\frac{2-x}{x^2-2x-8}=+\infty .$$

Another way to see this is to construct the sign table:

x	$-\infty$		-2		2		4		$+\infty$
$2-x$			+		0		-		
x^2-2x-8		+	0		-		0	+	
$\frac{2-x}{x^2-2x-8}$		+	|| ∞	-	0	+	|| ∞	-	

From the last row of the above table, we find that;

$$\lim_{x\to 4^+}\frac{2-x}{x^2-2x-8}=-\infty \text{ and } \lim_{x\to 4^-}\frac{2-x}{x^2-2x-8}=+\infty$$

Limits at infinity

Let us start by looking at **operations with infinity.**

a) Addition:

When you add two non-zero numbers you get a new number. For example $3+8=11$. But with infinity, this is not true. A really large positive number plus any positive number, regardless of size, is still a really large positive number.

With infinity you have the following:

$$+\infty+c=+\infty \quad with \quad c\neq-\infty$$

$$+\infty+\infty=+\infty$$

b) Subtraction:

A really large negative number minus any positive number, regardless of size, is still a really large negative number. In the case of subtraction we have;

$$-\infty-c=-\infty \; with \; c\neq-\infty$$

$$-\infty-\infty=-\infty$$

c) Multiplication:

A really large (positive or negative) number times any number, regardless of size, is still a really large number and we have to be careful with signs. In the case of multiplication, we have;

$$(a)(\infty)=\infty \; with \, a>0$$

$$(a)(\infty)=-\infty \; with \, a<0$$

$$(\infty)(\infty)=\infty$$

$$(-\infty)(-\infty)=\infty$$

$$(-\infty)(\infty)=-\infty$$

d) Division:

A really large (positive or negative) number divided by any number that is not too large, is still a really large number and we have to be careful with signs.

$$\frac{\infty}{a} = \infty \ if \ a > 0 \, and \ a \neq \infty$$

$$\frac{\infty}{a} = -\infty \ if \ a < 0 \, and \ a \neq -\infty$$

$$\frac{-\infty}{a} = -\infty \ if \ a > 0 \, and \ a \neq \infty$$

$$\frac{-\infty}{a} = \infty \ if \ a < 0 \, and \ a \neq -\infty$$

$$\frac{a}{\infty} = 0$$

$$\frac{a}{-\infty} = 0$$

Beware!

So, we have dealt with almost every basic algebraic operation involving infinity. There are three cases that we have not dealt with yet. These are $\infty - \infty = ?$, $0 \cdot \infty = ?$, $\frac{\infty}{\infty} = ?$

They are one that we call **indeterminate form** in limits calculation.

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$ and

$g(x) = b_m x^m + b_{m-1} x^{m-1} + ... + b_0$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left(a_n x^n + a_{n-1} x^{n-1} + ... + a_0 \right)$$

$$= \lim_{x \to \infty} a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \frac{a_{n-2}}{a_n x^2} + ... + \frac{a_0}{a_n x^n} \right)$$

$$= \lim_{x \to \infty} a_n x^n$$

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty}\left(b_m x^m + a_{m-1}x^{m-1} + \ldots + a_0\right)$$
$$= \lim_{x\to\infty} b_m x^m \left(1 + \frac{b_{m-1}}{b_m x} + \frac{b_{m-2}}{b_m x^2} + \ldots + \frac{b_0}{b_m x^m}\right)$$
$$= \lim_{x\to\infty} b_m x^m$$

And then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{a_n x^n}{b_m x^m}$$

We have three cases

a) If $m = n$, $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m}$.

b) If $n > m$, $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{a_n x^{n-m}}{b_m} = \infty$.

c) If $n < m$, $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{a_n}{b_m x^{m-n}} = 0$.

Example 17

$$\lim_{x\to+\infty}(-6) = -6$$

Example 18

$$\lim_{x\to-\infty}\left(3x^2 + 5x - 3\right) = \lim_{x\to-\infty} 3x^2$$
$$= +\infty$$

Example 19

Find the limit $\lim_{x\to-\infty}\frac{1}{x}$ and $\lim_{x\to+\infty}\frac{1}{x}$

Solution

From some sample calculations,

$\lim_{x\to-\infty}\frac{1}{x} = 0$ and $\lim_{x\to+\infty}\frac{1}{x} = 0$. We can write $\lim_{x\to\pm\infty}\frac{1}{x} = 0$.

Example 20

$$\lim_{x\to\infty}\frac{2x^4+3x^2+1}{3x^4+5x^2+3}=\lim_{x\to\infty}\frac{2x^4}{3x^4}$$
$$=\lim_{x\to\infty}\frac{2}{3}$$
$$=\frac{2}{3}$$

Example 21

$$\lim_{x\to-\infty}\frac{4x^3+5x-3}{x^2+3x+1}=\lim_{x\to-\infty}\frac{4x^3}{x^2}$$
$$=\lim_{x\to-\infty}4x$$
$$=-\infty$$

Example 22

$$\lim_{x\to\infty}\frac{5x+2}{3x^2+1}=\lim_{x\to\infty}\frac{5x}{3x^2}$$
$$=\lim_{x\to\infty}\frac{5}{3x}$$
$$=0$$

Exercise 4

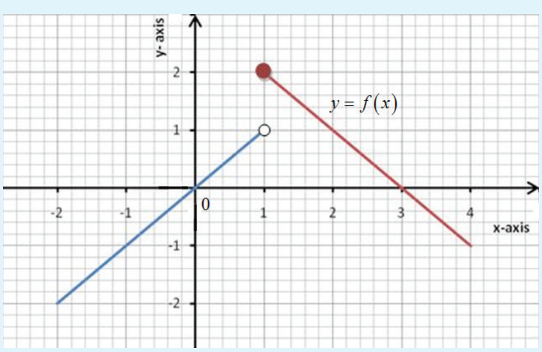
Evaluate the following limits:

1. $\lim_{x\to\infty}\frac{3x^3+2x^2-1}{6x^3+x+4}$
2. $\lim_{x\to-\infty}\frac{(x+3)^2}{x^3+4x^2-8x-4}$
3. $\lim_{x\to\infty}\frac{4x^3+x^2-1}{x^2-x+4}$
4. $\lim_{x\to-4}\frac{x+1}{x+4}$
5. $\lim_{x\to3}\frac{x^2+2x+1}{x-3}$

Finding limits graphically

Activity 5

Consider the following curve of function $f(x)$.



Use this graph to find:

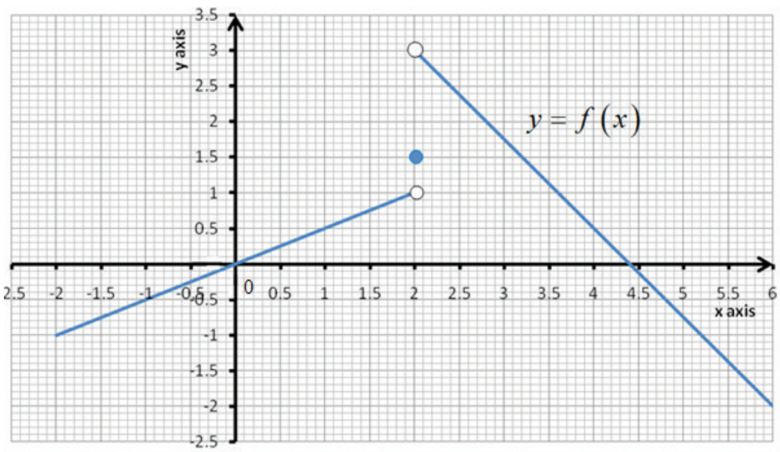
1. $\lim_{x\to1^-}f(x)$
2. $\lim_{x\to1^+}f(x)$

What can you say about $\lim_{x\to1}f(x)$?

To find a limit graphically, we must understand each component of the limit to ensure the graph is used properly to evaluate the limit.

Example 23

Let f be the function whose graph is shown below,

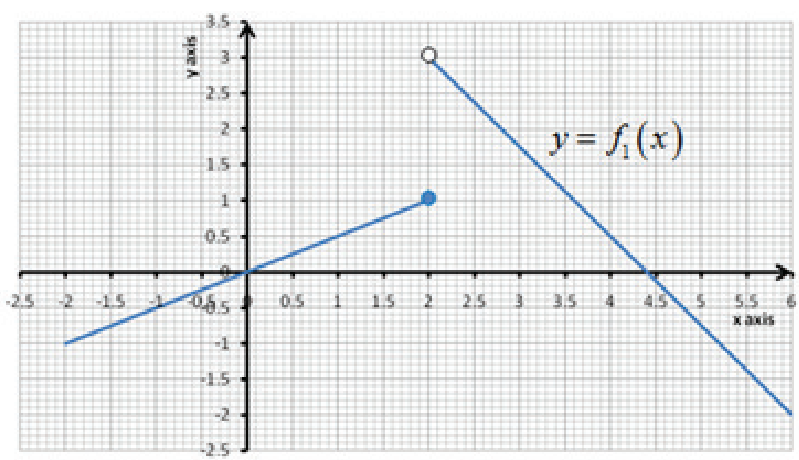


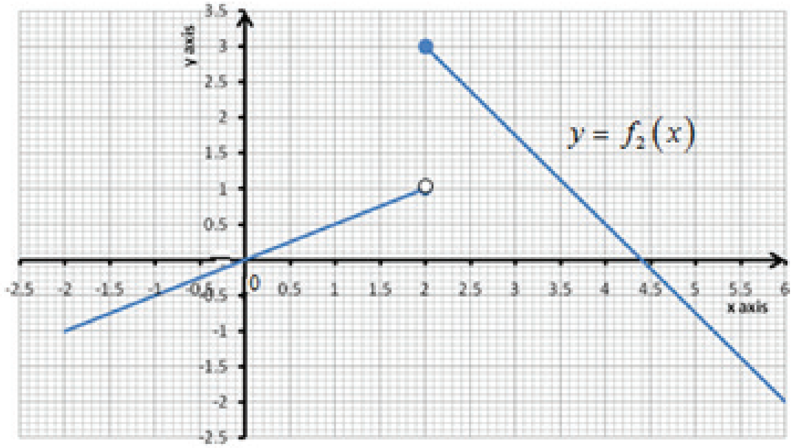
As x approaches 2 from the left, $f(x)$ approaches 1,

so $\lim_{x \to 2^-} f(x) = 1$. As x approaches 2 from the left, $f(x)$

approaches 3, so $\lim_{x \to 2^+} f(x) = 3$ but $f(2) = 1.5$

Therefore, the value of a function at a point, and the left and right hand limits at the point can all be different.





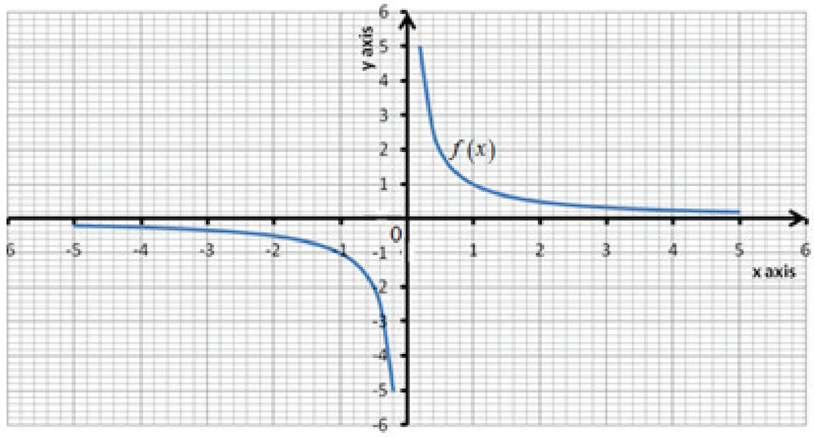
If we compare f_1, f_2 and f , we find that $f(2) = 1.5$ while

$f_1(2) = 1$ but $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} f_1(x) = \lim_{x \to 2^-} f_2(x) = 1$ and

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} f_1(x) = \lim_{x \to 2^+} f_2(x) = 3$.

Example 24

Let f be the function whose graph is:



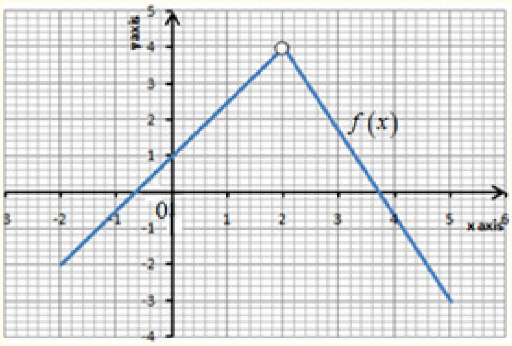
As x approaches 0 from the right side, $f(x)$ gets larger and larger without bound and consequently approaches no fixed value. Thus, $\lim_{x \to 0^+} f(x)$ **does not exist**. In this case, we would write $\lim_{x\to 0^+} f(x) = +\infty$ to indicate that the limit fails to exist because $f(x)$ is increasing without bound.

As x approaches 0 from the left side, $f(x)$ becomes more and more negative without bound and consequently approaches no fixed value. Thus, $\lim_{x\to 0^-} f(x)$ **does not exist**. In this case, we write $\lim_{x\to 0^-} f(x) = -\infty$ to indicate that the limit fails to exist because $f(x)$ is decreasing without bound.

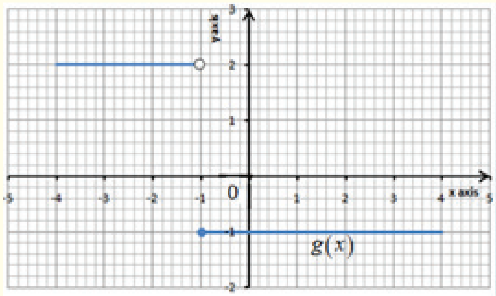
As x gets larger and larger, $f(x)$ gets close to zero. Also as x becomes more and more negative, $f(x)$ is close to zero. Thus, $\lim_{x\to +\infty} f(x) = 0$ and $\lim_{x\to -\infty} f(x) = 0$.

Exercise 5

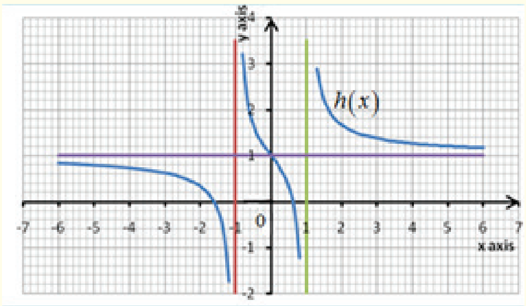
1. Find $\lim_{x\to 2} f(x)$ using the following graph of $f(x)$:



2. Find $\lim_{x\to -1} g(x)$ using the following graph of $g(x)$:



3. Find $\lim\limits_{x\to -1} h(x)$, $\lim\limits_{x\to 1} h(x)$, $\lim\limits_{x\to -\infty} h(x)$, $\lim\limits_{x\to \infty} h(x)$ using the following graph of $h(x)$:



The squeeze theorem and operations on limits

Activity 6

1. In the same Cartesian plane, sketch the curves of:

   $f(x)=x^2+5$, $g(x)=-x^2+5$ and $h(x)=5$. What can you say about the three curves?

   Evaluate $\lim\limits_{x\to 0} f(x)$, $\lim\limits_{x\to 0} g(x)$, $\lim\limits_{x\to 0} h(x)$

2. Evaluate the following limits

   a) $\lim\limits_{x\to 0}\left[3(3x-1)\right]$, $3\left[\lim\limits_{x\to 0}(3x-1)\right]$

   b) $\lim\limits_{x\to 0}\left(x^2\right)$, $\lim\limits_{x\to 0}(3x-1)$, $\lim\limits_{x\to 0}\left(x^2+3x-1\right)$

   c) $\lim\limits_{x\to 1}\left(x^2+3x-6\right)$, $\lim\limits_{x\to 1}(x+4)$, $\lim\limits_{x\to 1}\dfrac{x^2+3x-6}{x+4}$

   d) $\lim\limits_{x\to 2}(x-1), \lim\limits_{x\to 2}(x+4), \lim\limits_{x\to 2}(x^2+3x-4)$

   e) $\lim\limits_{x\to -4}\left[\left(x^2+1\right)^2\right]$, $\left[\lim\limits_{x\to -4}\left(x^2+1\right)\right]^2$

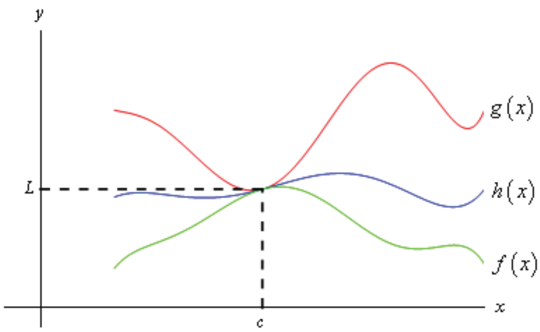
   What can you conclude?

The squeeze theorem (or Sandwich theorem or Pinching theorem)

Suppose that $f(x) \leq h(x) \leq g(x)$. If $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L$

then $\lim_{x \to c} h(x) = L$.

The following figure illustrates what is happening in this theorem:



From the figure, we can see that if the limits of $f(x)$ and $g(x)$ are equal at $x = c$ then the function values must also be equal at $x = c$. However, because $h(x)$ is "squeezed" between $f(x)$ and $g(x)$ at this point then $h(x)$ must have the same value. Therefore, the limit of $h(x)$ at this point must also be the same. Similar statements hold for left and right limits.

Example 25

Given that

$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}.$$

Find $\lim_{x \to 0} u(x)$

Solution

Since

$$\lim_{x \to 0}\left(1 - \frac{x^2}{4}\right) = 1 \quad and \quad \lim_{x \to 0}\left(1 + \frac{x^2}{2}\right) = 1$$

the Sandwich theorem implies that $\lim_{x \to 0} u(x) = 1$.

Example 26

Show that if

$\lim_{x \to a} |f(x)| = 0$ then $\lim_{x \to a} f(x) = 0$

Solution

Since $-|f(x)| \leq f(x) \leq |f(x)|$,

and $-|f(x)|$ and $|f(x)|$ both have limit 0 as x approaches a, so does $f(x)$ by the Sandwich theorem.

Operations on limits:

Let lim stand for one of the limits $\lim\limits_{x\to a}$, $\lim\limits_{x\to a^-}$, $\lim\limits_{x\to a^+}$, $\lim\limits_{x\to -\infty}$ or $\lim\limits_{x\to +\infty}$. If $\lim f(x)$ and $\lim g(x)$ both exist, say $\lim f(x) = L_1$ and $\lim g(x) = L_2$, then;

a) A constant factor can be moved through a limit sign. That is, if k is a constant, then $\lim\left[kf(x)\right] = k\lim f(x)$.

b) $\lim\left[f(x)+g(x)\right] = \lim f(x) + \lim g(x) = L_1 + L_2$.

c) $\lim\left[f(x)-g(x)\right] = \lim f(x) - \lim g(x) = L_1 - L_2$.

d) $\lim\left[f(x)\cdot g(x)\right] = \lim f(x)\cdot \lim g(x) = L_1 \cdot L_2$.

e) $\lim\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim f(x)}{\lim g(x)} = \dfrac{L_1}{L_2}$ if $L_2 \neq 0$.

f) If n and m are positive integers, then $\lim\left[f(x)\right]^{\frac{m}{n}} = L_1^{\frac{m}{n}}$ provided that $L_1 \geq 0$ if n is even.

Example 27

Find

$$\lim_{x\to 3} x^4 = \left[\lim_{x\to 3} x\right]^4 = 3^4 = 81$$

Solution

$$\lim_{x\to 3} x^4 = \left[\lim_{x\to 3} x\right]^4 = 3^4 = 81$$

Example 28

Find $\lim\limits_{x\to 5}\left(x^2 - 4x + 3\right)$

Solution

$$\begin{aligned}\lim_{x\to 5}\left(x^2-4x+3\right) &= \lim_{x\to 5} x^2 - \lim_{x\to 5} 4x + \lim_{x\to 5} 3\\ &= \lim_{x\to 5} x^2 - 4\lim_{x\to 5} x + \lim_{x\to 5} 3\\ &= 5^2 - 4\lim_{x\to 5} x + \lim_{x\to 5} 3\\ &= 25 - 4(5) + 3\\ &= 8\end{aligned}$$

Example 29

Find $\lim\limits_{x\to 2}\dfrac{f(x)}{g(x)}$ if $f(x)=5x^3+4$

and $g(x)=x-3$

Solution

$$\lim_{x\to 2}\frac{f(x)}{g(x)}=\lim_{x\to 2}\frac{5x^3+4}{x-3}$$

$$\lim_{x\to 2}\frac{5x^3+4}{x-3}=\frac{\lim\limits_{x\to 2}(5x^3+4)}{\lim\limits_{x\to 2}(x-3)}$$

$$=\frac{5(2)^3+4}{2-3}$$

$$=-44$$

Example 30

Find $\lim\limits_{x\to 0}f(x)g(x)$ if $f(x)=6x^2+2$ and $g(x)=x+2$

Solution

$$\lim_{x\to 0}f(x)g(x)$$

$$=\lim_{x\to 0}(6x^2+2)\lim_{x\to 0}(x+2)=2\times 2=4$$

Exercise 6

1. Given that $-x^2\le g(x)\le x^2$. Find $\lim\limits_{x\to 0}g(x)$.

2. If $\lim\limits_{x\to 3}f(x)=6$ and $\lim\limits_{x\to 3}g(x)=-3$. Find;

   a) $\lim\limits_{x\to 3}\left[f(x)+g(x)\right]$ b) $\lim\limits_{x\to 3}\left[f(x)g(x)\right]^3$

3. Explain why the following calculation is incorrect:

   a) $\lim\limits_{x\to 0^+}\left(\dfrac{1}{x}-\dfrac{1}{x^2}\right)=\lim\limits_{x\to 0^+}\dfrac{1}{x}-\lim\limits_{x\to 0^+}\dfrac{1}{x^2}=\infty-\infty=0$

   b) Show that $\lim\limits_{x\to 0^+}\left(\dfrac{1}{x}-\dfrac{1}{x^2}\right)=-\infty$

2. Indeterminate cases

Activity 7

1. Find a common factor for numerator and denominator.

a) $f(x)=\dfrac{x^2-1}{x-1}$ b) $f(x)=\dfrac{x^3+x^2-5x-2}{x^2-4}$

An indeterminate form is a certain type of expression with a limit that is not evident by inspection. There are several types of indeterminate forms such as $\frac{0}{0},\frac{\infty}{\infty},0\cdot\infty,\infty-\infty,0^0,1^\infty$

In this section we will study the forms $\frac{0}{0},\frac{\infty}{\infty},0\cdot\infty,\infty-\infty$.

The indeterminate forms may be produced in the following ways:

- Suppose that $\lim\limits_{x\to a} f(x)=0$ and $\lim\limits_{x\to a} g(x)=\infty$.

The limit of the product $f(x)g(x)$ has the indeterminate form, $0\bullet\infty$, at $x=a$.

To evaluate this limit we try to change the limit into one of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ in this way: $f(x)g(x)=\dfrac{f(x)}{\frac{1}{g(x)}}=\dfrac{g(x)}{\frac{1}{f(x)}}$

- If $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} g(x)=+\infty$, then $\lim\limits_{x\to a}\left[f(x)-g(x)\right]$ has the indeterminate form $\infty-\infty$. To evaluate this limit, we try the algebraic manipulations by converting the limit into a form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$. If $f(x)$ or $g(x)$ is expressed as a fraction, we can do this by finding the common denominator.

Example 31

Evaluate

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

Solution

$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \frac{0}{0}$, this is the indeterminate form.

By factoring the numerator and cancelling the common factor, we obtain;

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2}$$
$$= \lim_{x \to 2} (x+2)$$
$$= 4$$

Example 32

Evaluate $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

Solution

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{1-1}{1-1} = \frac{0}{0} I.F$$

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{x-1}$$
$$= \lim_{x \to 1} (x+1)$$
$$= 2$$

Example 33

Evaluate $\lim_{x \to +\infty} \frac{x^2 + 4x + 5}{4x^2 + 7x + 9}$

Solution

$$\lim_{x \to +\infty} \frac{x^2 + 4x + 5}{4x^2 + 7x + 9} = \frac{\infty}{\infty} I.F$$

Factor out x^2, we have;

$$\lim_{x \to +\infty} \frac{x^2\left(1 + \frac{4}{x} + \frac{5}{x^2}\right)}{x^2\left(4 + \frac{7}{x} + \frac{9}{x^2}\right)} = \lim_{x \to +\infty} \frac{1 + \frac{4}{x} + \frac{5}{x^2}}{4 + \frac{7}{x} + \frac{9}{x^2}}$$
$$= \frac{1+0+0}{4+0+0}$$
$$= \frac{1}{4}$$

Or Since we have a rational function and degree of numerator is equal to the degree of denominator, to find the limit as x tends to infinity, we need to divide the coefficients of the highest degree for numerator and denominator. That is the limit is given by $\frac{1}{4}$.

Example 34

Evaluate $\lim_{x \to -\infty} (3x^2 + 5x - 3)$

Solution

$$\lim_{x\to-\infty}\left(3x^2+5x-3\right)=3\left(-\infty\right)^2+5\left(-\infty\right)-3$$
$$=+\infty-\infty\ I.F$$

Factor out x^2:

$$\lim_{x\to-\infty}\left(3x^2+5x-3\right)=\lim_{x\to-\infty}x^2\left(3+\frac{5}{x}-\frac{3}{x^2}\right)$$
$$=+\infty\left(3+0-0\right)$$
$$=+\infty$$

Exercise 7

Evaluate the following limits

1. $\lim_{x\to\infty}\left(x^2-2x+5\right)$
2. $\lim_{x\to-\infty}\left(4x^3+3x^2-6\right)$
3. $\lim_{x\to4}\frac{x^4-16}{x^2-4}$

Indeterminate forms in irrational functions

Activity 8

What is the conjugate of the irrational expression in each of the following functions?

a) $f\left(x\right)=\sqrt{x^2-2}+3$

b) $f\left(x\right)=\frac{\sqrt{x-2}-1}{x-3}$

When we are computing the limits of irrational functions, in case of indeterminate form, we need to know the conjugate of the irrational expression in that function. We may need to find the domain of the given function.

Example 35

Evaluate $\lim_{x\to4}\frac{\sqrt{x-3}-1}{x-4}$.

Solution

$$\lim_{x\to 4}\frac{\sqrt{x-3}-1}{x-4}=\frac{1-1}{4-4}=\frac{0}{0}\,I.F$$

To evaluate this limit, we multiply the numerator and denominator by the conjugate of $\sqrt{x-3}-1$ which is $\sqrt{x-3}+1$, then

$$\lim_{x\to 4}\frac{\sqrt{x-3}-1}{x-4}=\lim_{x\to 4}\frac{\left(\sqrt{x-3}-1\right)\left(\sqrt{x-3}+1\right)}{(x-4)\left(\sqrt{x-3}+1\right)}$$

$$=\lim_{x\to 4}\frac{x-3-1}{(x-4)\left(\sqrt{x-3}+1\right)}$$

$$=\lim_{x\to 4}\frac{1}{\sqrt{x-3}+1}$$

$$=\frac{1}{2}$$

Example 36

Evaluate $\lim_{x\to+\infty}\left(\sqrt{4x^2+2}-2x\right)$

Solution

$$\lim_{x\to+\infty}\left(\sqrt{4x^2+2}-2x\right)=+\infty-\infty\ \ I.F$$

To evaluate this limit we multiply and divide by the conjugate of $\sqrt{4x^2+2}-2x$.

$$\lim_{x\to+\infty}\left(\sqrt{4x^2+2}-2x\right)=\lim_{x\to+\infty}\frac{\left(\sqrt{4x^2+2}-2x\right)\left(\sqrt{4x^2+2}+2x\right)}{\sqrt{4x^2+2}+2x}$$

$$=\lim_{x\to+\infty}\frac{4x^2+2-4x^2}{\sqrt{4x^2+2}+2x}$$

$$=\lim_{x\to+\infty}\frac{2}{\sqrt{4x^2+2}+2x}$$

$$=\frac{2}{+\infty}$$

$$=0$$

Example 37

Evaluate $\lim_{x\to+\infty}\frac{\sqrt{4x^2-11x-3}}{x}$

Solution

$$\lim_{x\to+\infty}\frac{\sqrt{4x^2-11x-3}}{x}=\frac{\sqrt{\infty-\infty}}{\infty}I.F$$

To evaluate this limit, we try the algebraic manipulations such that the denominator will be cancelled.

$$\lim_{x\to+\infty}\frac{\sqrt{4x^2-11x-3}}{x}=\lim_{x\to+\infty}\frac{\sqrt{4x^2\left(1-\frac{11}{4x}-\frac{3}{4x^2}\right)}}{x}$$

$$=\lim_{x\to+\infty}\frac{\left(\sqrt{4x^2}\right)\sqrt{1-\frac{11}{4x}-\frac{3}{4x^2}}}{x}$$

$$=\lim_{x\to+\infty}\frac{\sqrt{4x^2}}{x}\lim_{x\to+\infty}\sqrt{1-\frac{11}{4x}-\frac{3}{4x^2}}$$

$$=\left(\lim_{x\to+\infty}\frac{\sqrt{4x^2}}{x}\right)\times 1$$

$$=\lim_{x\to+\infty}\frac{\sqrt{4x^2}}{x}$$

$$=\lim_{x\to+\infty}\frac{2\sqrt{x^2}}{x}$$

Recall that $\sqrt{x^2}=\begin{cases}x & if\ \ x\geq 0\\ -x & if\ \ x<0\end{cases}$

We need to find the domain of the given function:

$Domf=\left]-\infty,-\frac{1}{4}\right]\cup[3,+\infty[$. As x tends to $+\infty$, $x\in[3,+\infty[$ and then $\sqrt{x^2}=x$.

Thus,

$$\lim_{x\to+\infty}\frac{\sqrt{4x^2-11x-3}}{x}=\lim_{x\to+\infty}\frac{2\sqrt{x^2}}{x}$$

$$=\lim_{x\to+\infty}\frac{2x}{x}$$

$$=2$$

Example 38

Evaluate

$$\lim_{x\to1}\frac{\sqrt[3]{x}-1}{x-1}$$

Solution

$$\lim_{x\to1}\frac{\sqrt[3]{x}-1}{x-1}=\frac{0}{0}\ I.F$$

$$\lim_{x\to1}\frac{\sqrt[3]{x}-1}{x-1}=\lim_{x\to1}\frac{\left(\sqrt[3]{x}-1\right)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)}{(x-1)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)}$$

$$=\lim_{x\to1}\frac{(x-1)}{(x-1)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)}$$

$$=\lim_{x\to1}\frac{1}{\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)}$$

$$=\frac{1}{3}$$

Example 39

Evaluate

$$\lim_{x\to2}\frac{\sqrt{x-2}}{\sqrt[3]{x-2}}$$

Solution

$$\lim_{x\to2}\frac{\sqrt{x-2}}{\sqrt[3]{x-2}}=\frac{0}{0}\ I.F$$

$$\lim_{x\to2}\frac{\sqrt{x-2}}{\sqrt[3]{x-2}}=\lim_{x\to2}\frac{(x-2)^{\frac{1}{2}}}{(x-2)^{\frac{1}{3}}}$$

$$=\lim_{x\to2}(x-2)^{\frac{1}{2}}(x-2)^{-\frac{1}{3}}$$

$$=\lim_{x\to2}(x-2)^{\frac{1}{2}-\frac{1}{3}}$$

$$=\lim_{x\to2}(x-2)^{\frac{1}{6}}$$

$$=\lim_{x\to2}\sqrt[6]{x-2}$$

$$=0$$

Note that the limits involving indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ can be evaluated by successive derivatives of numerator and denominator. This method is called **L'Hôpital rule**.

We will see this in application of derivatives.

Exercise 8

Evaluate the following limits;

1. $\lim\limits_{x\to 4}\frac{\sqrt{x^2-6}-10}{x-4}$

2. $\lim\limits_{x\to -\infty}\frac{3x}{\sqrt{9x^2-3x+6}}$

3. Applications

Continuity of function

Activity 9

Given the function $f(x)=\begin{cases}\frac{x^2-4}{x-2}, & x\neq 2\\ 4, & x=2\end{cases}$, find;

a) $f(2)$ b) $\lim\limits_{x\to 2}f(x)$

What can you say about $f(2)$ and $\lim\limits_{x\to 2}f(x)$?

a) Continuity of a function at a point or on interval I

A function $f(x)$ is said to be **continuous at point c** if the following conditions are satisfied:

a) $f(c)$ is defined b) $\lim\limits_{x\to c}f(x)$ exists c) $\lim\limits_{x\to c}f(x)=f(c)$

If one or more conditions in this definition fail to hold, then f is said to be **discontinuous at point c**, and c is called **a point of discontinuity** of f. If f is continuous at all point of an open interval (a,b), then f is said to be continuous on (a,b).

A function that is continuous on $(-\infty,+\infty)$ is said to be **continuous everywhere** or simply **continuous**.

Example 40

Study the continuity of $f(x)=\frac{x^2-4}{x-2}$ at $x=2$

Solution

The function $f(x)=\dfrac{x^2-4}{x-2}$ is discontinuous at 2 because $f(2)$ is undefined.

Example 41

Study the continuity of $g(x)=\begin{cases}\dfrac{x^2-4}{x-2} & if\ \ x\neq 2\\ 3 & if\ \ x=2\end{cases}$ at $x=2$

Solution

The function $g(x)=\begin{cases}\dfrac{x^2-4}{x-2} & if\ \ x\neq 2\\ 3 & if\ \ x=2\end{cases}$ is discontinuous at 2

because $g(2)=3$ while $\lim\limits_{x\to 2} g(x)=\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}=\lim\limits_{x\to 2}(x+2)=4$

so that $\lim\limits_{x\to 2} g(x)\neq g(2)$.

Example 42

Study the continuity of $g(x)=\begin{cases}\dfrac{x^2-9}{x-3} & if\ \ x\neq 3\\ 6 & if\ \ x=3\end{cases}$ at $x=3$

Solution

The function $g(x)=\begin{cases}\dfrac{x^2-9}{x-3} & if\ \ x\neq 3\\ 6 & if\ \ x=3\end{cases}$ is continuous at 3

since

$g(3)=6$ and $\lim\limits_{x\to 3^+} g(x)=\lim\limits_{x\to 3^-} g(x)=\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}=\lim\limits_{x\to 3}(x+3)=6$

so that $\lim\limits_{x\to 3} g(x)=g(3)$.

Example 43

Study the continuity of at $f(x)=\cos x$ at $x=0$

Solution

The function $f(x)=\cos x$ is continuous at 0 because $f(0)=1$

and $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\cos x=\cos 0=1$ so that $\lim\limits_{x\to 0} f(x)=f(0)$.

Example 44

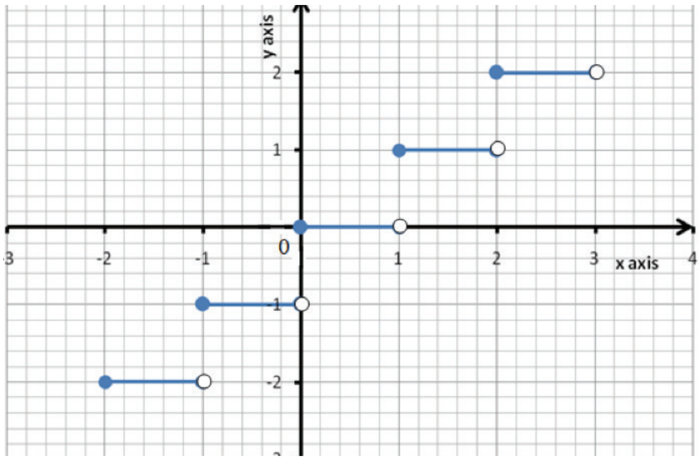
The function $f: x \to [x]$, where $[x]$ is used to denote the greatest integer less or equal to x, has a discontinuity at each integral value of x because, for example, where $x = 2$

: $f(2) = [2] = 2$ and $\lim_{x \to 2^+} [x] = 2, \lim_{x \to 2^-} [x] = 1$ so that $\lim_{x \to 2} f(x)$ does not exist.

If $x = 2.4, [x] = 2$

If $x = -0.8, [x] = -1$

If $x = 4, [x] = 4$



Notice

The function $[x]$ is also denoted $\lfloor x \rfloor$ or $[\![x]\!]$.

$\lceil x \rceil$ or $]x[$ is used to denote the least integer greater than or equal to x

b) Continuity at the left and continuity at the right of a point

A function f is continuous at the left of point c if the following conditions are satisfied:

a) $f(c)$ is defined b) $\lim_{x \to c^-} f(x)$ exists c) $\lim_{x \to c^-} f(x) = f(c)$

A function f is continuous at the right at of point c if the following conditions are satisfied:

a) $f(c)$ is defined b) $\lim_{x \to c^+} f(x)$ exists c) $\lim_{x \to c^+} f(x) = f(c)$

Example 45

From example 5, $\lim_{x\to 2^+}[2]=2=[2]$. Therefore $[x]$ is continuous at the right of point 2.

c) Continuity at an endpoint

We say that *f* is continuous at a left endpoint of its domain if it is continuous at the right of that point.

We say that *f* is continuous at a right endpoint of its domain if it is continuous at the left of that point.

Example 46

The function $f(x)=\sqrt{4-x^2}$ has domain $[-2,2]$

This function is continuous at the right endpoint 2 because it is left continuous there, that is, because $\lim_{x\to 2^-} f(x)=0=f(2)$.

This function is continuous at the left endpoint -2 because it is right continuous there, that is, because $\lim_{x\to 2^+} f(x)=0=f(-2)$.

d) Continuity on an interval

We say that *f* is **continuous on the interval *I*** if it is continuous at each point of *I*. In particular, we will say that *f* is a continuous function if *f* is continuous at every point of its domain.

Example 47

The function $f(x)=\sqrt{x}$ is a continuous function on its domain. Its domain is $[0,+\infty[$. It is continuous at the left endpoint 0 because it is right continuous there. Also, f is continuous at every number $c>0$ since $\lim_{x\to c}\sqrt{x}=\sqrt{c}$.

Example 48

The function $f(x)=\frac{x}{\sqrt{1-x^2}}$ is a continuous function on its domain. Its domain is]-1,1[. It is continuous at the left endpoint -1 because it is right continuous there. It is continuous at the right endpoint 1 because it is left continuous there. Also, f is continuous at every number $c \in$]-1,1[since $\lim\limits_{x\to c}\frac{x}{\sqrt{1-x^2}}=\frac{c}{\sqrt{1-c^2}}$.

e) Continuity on a closed interval

A function *f* is continuous on a closed interval $[a,b]$, if the following conditions are all satisfied;

a) *f* is continuous on (a,b).
b) *f* is continuous at the right of *a*.
c) *f* is continuous at the left of *b*.

Example 49

The function $f(x)=\sqrt{9-x^2}$ is continuous on the closed interval $[-3,3]$.

We observe that $Domf=[-3,3]$. For *c* in the interval $(-3,3)$ we have $\lim\limits_{x\to c}f(x)=\lim\limits_{x\to c}\sqrt{9-x^2}=\sqrt{9-c^2}=f(c)$. So that *f* is continuous on $(-3,3)$.

Also, $\lim\limits_{x\to 3^-}f(x)=\lim\limits_{x\to 3^-}\sqrt{9-x^2}=0=f(3)$ and

$\lim\limits_{x\to -3^+}f(x)=\lim\limits_{x\to -3^+}\sqrt{9-x^2}=0=f(-3)\cdot$

So that *f* is continuous on $[-3,3]$.

Theorem

a) Polynomials are continuous functions.
b) If the functions *f* and *g* are continuous at *c*, then;

(i) $f+g$ is continuous at *c*.

(ii) $f-g$ is continuous at c.

(iii) $f \cdot g$ is continuous at c.

(iv) $\frac{f}{g}$ is continuous at c if $g(c) \neq 0$, and is discontinuous at c if $g(c)=0$.

c) A rational function is continuous everywhere except at the point where the denominator is zero.

d) Piecewise functions (functions that are defined on a sequence of intervals) are continuous if every function is in its interval of definition, and if the functions match their side limits at the points of separation of their intervals.

Example 50

The function $h(x)=\frac{x^2-9}{x^2-5x+6}$ is continuous everywhere except at point 2 and 3 because the numerator and denominator are polynomials and denominator is zero at points 2 and 3.

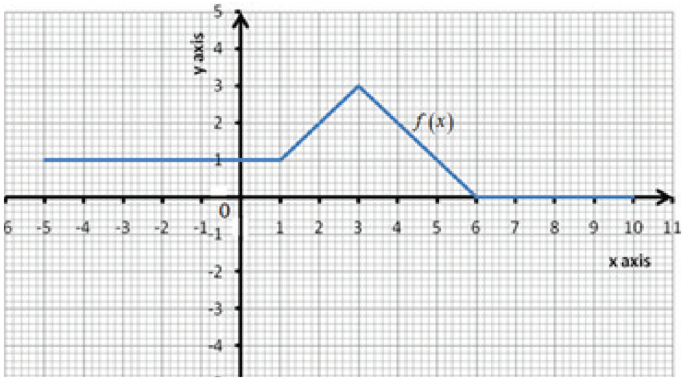
Example 51

The function

$$f(x)=\begin{cases} 1 & \text{if } x \leq 1 \\ x & \text{if } 1 < x \leq 3 \\ -x+6 & \text{if } 3 < x \leq 6 \\ 0 & \text{if } x > 6 \end{cases}$$

is continuous on $\mathbb{R}$, because its constituent functions are polynomials and the side limits at the points of division coincide.

Here is the graph



Exercise 9

1. Determine where the function below is not continuous.

$$f(x)=\frac{4x+10}{x^2-2x-15}$$

2. Given the function: $f(x)=\begin{cases}\dfrac{x^2-9}{x-3}, & x\neq 3\\ k, & x=3\end{cases}$

Determine the value of k for which the function is continuous at $x = 3$.

3. Determine the values for a and b in order to create a continuous function.

$$f(x)=\begin{cases}\dfrac{1}{x^2+1}, & x<0\\ ax+b, & 0\leq x\leq 3\\ x-5, & x>3\end{cases}$$

Classification of discontinuity

Activity 10

Evaluate

1. $\lim\limits_{x\to 3^-} f(x),\ \lim\limits_{x\to 3^+} f(x)\quad for\ f(x)=\begin{cases}x+1, & x>3\\ x^2, & x\leq 3\end{cases}$

2. $\lim\limits_{x\to 5} f(x),\quad f(x)=\dfrac{x^2-7x+10}{x-5}$

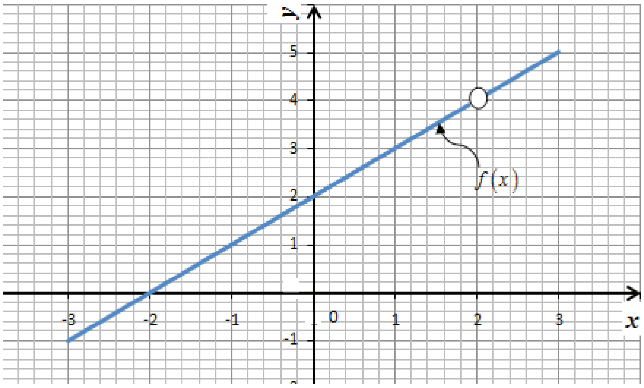
a) Apparent (or removable) discontinuity and continuous extensions

As we have seen, a rational function may have a limit even at a point where its denominator is zero. If $f(x)$ is undefined at a but $\lim_{x\to a} f(x) = L$ exists, we can define a new function $F(x)$ by;

$$F(x) = \begin{cases} f(x) & \text{if } x \text{ is in the domain of } f \\ L & \text{if } x = a \end{cases}$$

$F(x)$ is continuous at $x = a$. It is called the **continuous extension** of $f(x)$ to $x = a$.

If a function is undefined or discontinuous at a point a but can be redefined at that single point so that it becomes continuous there, then we say that f has a **removable** (or **apparent**) discontinuity at a.



Example 52

The function $f(x) = \frac{x^2 - 4}{x - 2}$ is continuous everywhere except at $x = 2$.

$f(x)$ is undefined at $x = 2$ but $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2}(x+2) = 4$ so that 2 is the apparent discontinuity point of $f(x) = \frac{x^2-4}{x-2}$.

Now the continuous extension of $f(x)$ is

$$f(x)=\begin{cases}\dfrac{x^2-4}{x-2} & \text{if } x\neq 2\\ 4 & \text{if } x=2\end{cases}$$

b) Discontinuity of the first kind or jump discontinuity

If $\lim\limits_{x\to c^+} f(x)$ and $\lim\limits_{x\to c^-} f(x)$ exist but are different, $f(x)$ is said to have a discontinuity of the first kind at point c and c is called discontinuity point of the first kind.

Example 53

Let $f(x)=\dfrac{x}{|x|}$

$f(x)$ is undefined at $x=0$, then 0 is a point of discontinuity.

$\lim\limits_{x\to 0^-}\dfrac{x}{|x|}=\lim\limits_{x\to 0}\dfrac{x}{-x}=-1$ and $\lim\limits_{x\to 0^+}\dfrac{x}{|x|}=\lim\limits_{x\to 0}\dfrac{x}{x}=1$.

As $\lim\limits_{x\to 0^+} f(x)$ and $\lim\limits_{x\to 0^-} f(x)$ exist but are different, there is a discontinuity point of the first kind at $x=0$.

c) Discontinuity of the second kind

A function f is said to have a discontinuity of the second kind at point c if atleast one of its limits from the left or from the right does not exist or is infinity.

Example 54

The function $f(x)=\dfrac{1}{x}$ has a discontinuity of the second kind at 0 because $\lim\limits_{x\to 0^+}\dfrac{1}{x}=+\infty$ and $\lim\limits_{x\to 0^-}\dfrac{1}{x}=-\infty$.

Exercise 10

Classify the points of discontinuity in each of the following functions

1. $f(x)=\frac{x^3+x^2-2x-2}{x+1}$

2. $f(x)=\begin{cases}2, & x\le 4\\ -4, & x>4\end{cases}$

3. $f(x)=\frac{x^2-2x-2}{x^2+2x+1}$

4. $f(x)=\frac{|x+2|}{x+2}$

5. $f(x)=\begin{cases}x+3, & x>-2\\ x^2-3, & x\le -2\end{cases}$

Theorem: Intermediate value theorem

Activity 11

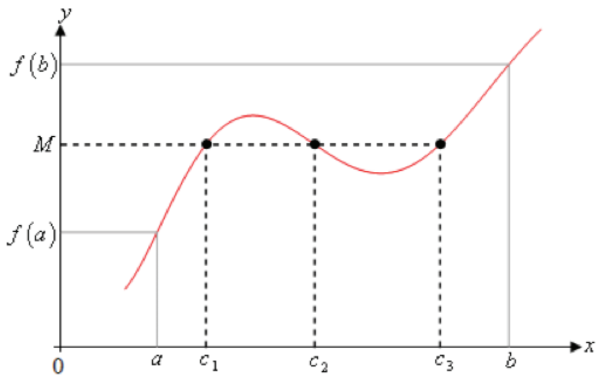
For each of the following functions, find two numbers a and b such that $f(a)f(b)<0$

1. $f(x)=x^3$ on interval $[-1,2]$
2. $f(x)=\frac{x^2-8}{x-3}$ on interval $[-5,0]$
3. $f(x)=2x^3-3x^2-12x+20$ on interval $[-4,-1]$

The Intermediate Value Theorem states that if we have a continuous function $f(x)$ on the interval $[a,b]$ with M being any number between $f(a)$ and $f(b)$, there exists a number c such that:

a) $a<c<b$ b) $f(c)=M$

The Intermediate Value Theorem is a geometrical application illustrating that continuous functions will take on all values between $f(a)$ and $f(b)$.



It is important to note that this theorem does not tell us the value of *M*, but only that it exists. For example, we can use this theorem to see if a function will have any *x*-intercepts.

Example 55

Use the Intermediate Value Theorem to determine if the function $f(x)=2x^3-5x^2-10x+5$ has a root somewhere in the interval $[-1,2]$.

In other words, we are asking if $f(x)=0$ in the interval $[-1,2]$.

Using the Intermediate Value Theorem, we can say that we want to show that there is a number c where $-1<c<2$ such that $f(c)=0$ between $f(-1)$ and $f(2)$.

We see that $f(-1)=8$ and $f(2)=-19$.

Therefore, $8>0>-19$ and atleast one root exists for $f(x)$ in the given interval.

Example 56

Show that there is a root of the equation $x^3-x-1=0$ between 1 and 2.

Let $f(x)=x^3-x-1$. Since $f(1)=-1<0$ and $f(2)=5>0$, we see that 0 is a value between $f(1)$ and $f(2)$. Since *f* is continuous, the Intermediate Value Theorem says that there is a zero of *f*(*x*) between 1 and 2.

Exercise 11

1. Use the Intermediate Value Theorem to determine if the function $y=-x^3+3x$ has a root somewhere in;

   a) the interval $[-3,0]$.   b) the interval $[0,4]$

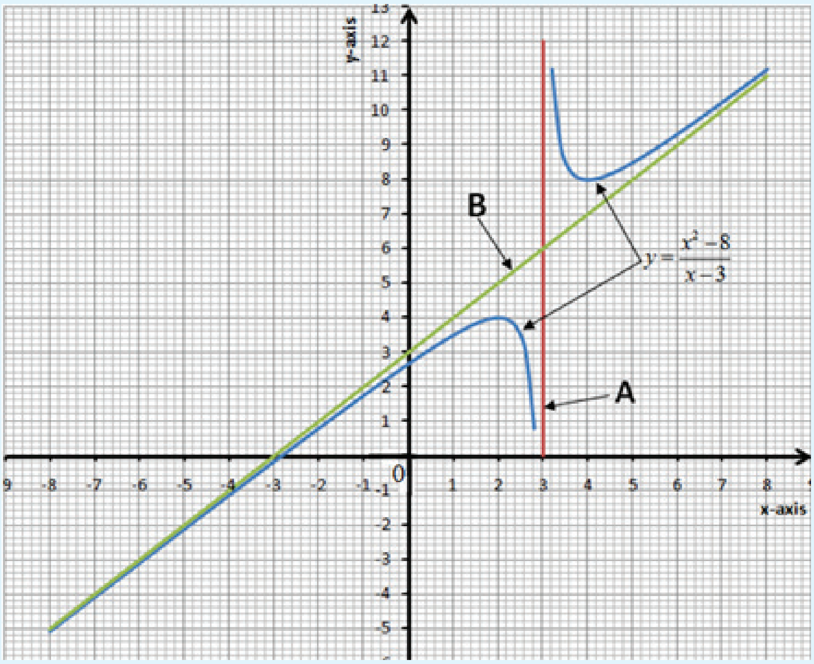
2. Use the Intermediate Value Theorem to determine if the function $y=\frac{2x}{x+1}$ has a root somewhere in the interval $[-2,4]$.

3. Use the Intermediate Value Theorem to determine if the function $y=\sqrt[3]{(x-1)^2(x+1)}$ has a root somewhere in the interval $[-4,1]$.

Asymptotes

Activity 12

Consider the following curve of function y



What can you say about the curve of y and the lines A and B?

Recall that if $P(x)$ and $Q(x)$ are polynomials, then their ratio $f(x)=\frac{P(x)}{Q(x)}$ is called a rational function of x. The discontinuity occurs at points where $Q(x)=0$.

An asymptote on the curve is a straight line that is closely approached by that curve so that the perpendicular distance between them decreases to zero.

To find any asymptote of the function, first we need to determine its domain of definition and evaluate the limits at the boundaries of the domain.

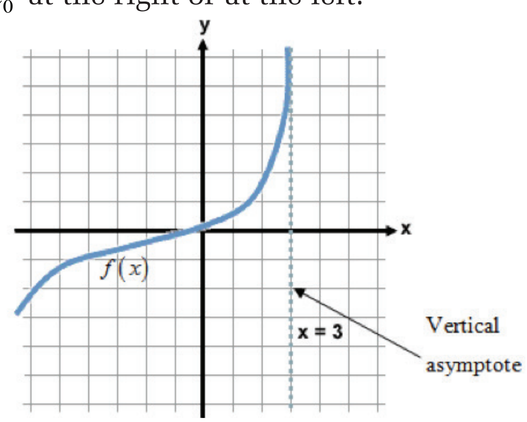
Types of asymptotes

There are three types of asymptotes:

- Vertical asymptote,
- Horizontal asymptote and
- Oblique asymptote.

a) Vertical asymptote

A line $x=x_0$ is called a **vertical asymptote** for the graph of a function $f(x)$ if $f(x)\to+\infty$ or $f(x)\to-\infty$ as x approaches x_0 at the right or at the left.



Example 57

Find the vertical asymptote for $f(x)=$

$$f(x)=\frac{2x^2+7x-1}{x+1}$$

Solution

Let $f(x)=\dfrac{2x^2+7x-1}{x+1}$

$Domf =]-\infty,-1[\cup]-1,+\infty[$

$$\lim_{x\to -1} f(x)=\lim_{x\to -1}\frac{2x^2+7x-1}{x+1}$$

$$=\infty$$

Hence, $x=-1$ is a vertical asymptote for $f(x)=\dfrac{2x^2+7x-1}{x+1}$.

Example 58

Find the vertical asymptote for

$$g(x)=\frac{x^3-2x-4}{4-x^2}$$

Solution

Let $g(x)=\dfrac{x^3-2x-4}{4-x^2}$

$Domg=\mathbb{R}\setminus\{-2,2\}$

$$\lim_{x\to -2} g(x)=\lim_{x\to -2}\frac{x^3-2x-4}{4-x^2}$$

$$=\infty$$

Hence, $x=-2$ is a vertical asymptote for $g(x)=\dfrac{x^3-2x-4}{4-x^2}$.

$$\lim_{x\to 2} g(x)=\lim_{x\to 2}\frac{x^3-2x-4}{4-x^2}$$

$$=\frac{0}{0}\ I.F$$

$$\lim_{x\to 2} g(x)=\lim_{x\to 2}\frac{(x-2)(x^2+2x+2)}{-(x-2)(x+2)}$$

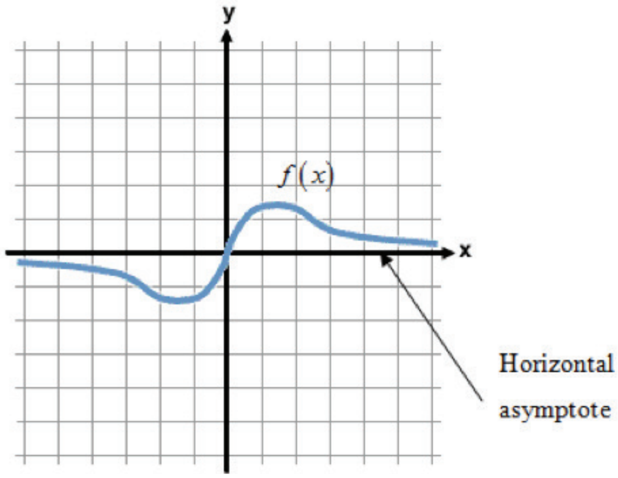
$$=\lim_{x\to 2}\frac{x^2+2x+2}{-(x+2)}$$

$$=-\frac{5}{2}$$

Since $\lim_{x\to 2} g(x)\neq\infty$, $x=2$ is not a vertical asymptote for $g(x)=\dfrac{x^3-2x-4}{4-x^2}$.

b) Horizontal asymptote

A line $y = L$ is called a **horizontal asymptote** for the graph of a function $f(x)$ if $f(x) \to L$ as $x \to +\infty$ or $x \to -\infty$.



Example 59

Find the horizontal asymptote for $f(x) = \dfrac{3x^2 + 4x - 9}{2x^2 + 1}$

Solution

Let $f(x) = \dfrac{3x^2 + 4x - 9}{2x^2 + 1}$

$Domf =]-\infty, +\infty[$

$\lim\limits_{x \to \pm\infty} f(x) = \lim\limits_{x \to \pm\infty} \dfrac{3x^2}{2x^2}$

$= \dfrac{3}{2}$

Thus, $y = \dfrac{3}{2}$ is a horizontal asymptote for

$f(x) = \dfrac{3x^2 + 4x - 9}{2x^2 + 1}$

Example 60

Find the horizontal asymptote for

$$h(x)=\frac{x\sqrt{4x^2+3x-1}}{x^2+5}$$

Solution

Let $h(x)=\frac{x\sqrt{4x^2+3x-1}}{x^2+5}$

$Domh=\left]-\infty,-1\right]\cup\left[\frac{1}{4},+\infty\right[$

$$\lim_{x\to-\infty} h(x)=\lim_{x\to-\infty}\frac{x\sqrt{4x^2+3x-1}}{x^2+5}$$
$$=-2$$

Hence, $y=-2$ is a horizontal asymptote for $h(x)=\frac{x\sqrt{4x^2+3x-1}}{x^2+5}$.

$$\lim_{x\to+\infty} h(x)=\lim_{x\to+\infty}\frac{x\sqrt{4x^2+3x-1}}{x^2+5}$$
$$=2$$

Hence, $y=2$ is another horizontal asymptote for

$$h(x)=\frac{x\sqrt{4x^2+3x-1}}{x^2+5}.$$

c) Oblique asymptote

If a rational function, $\frac{P(x)}{Q(x)}$, is such that the degree of the numerator exceeds the degree of the denominator by one, then the graph of $\frac{P(x)}{Q(x)}$ will have an **oblique asymptote (or a slant asymptote)**; that is, an asymptote that is neither vertical nor horizontal.

We perform the division of $P(x)$ by $Q(x)$ to obtain

$$\frac{P(x)}{Q(x)}=(ax+b)+\frac{R(x)}{Q(x)}$$

Where, $ax+b$ is the quotient and $R(x)$ is the remainder.

Use the fact that the degree of the remainder $R(x)$ is less than the degree of the divisor $Q(x)$ to help prove:

$$\lim_{x\to+\infty}\left|\frac{P(x)}{Q(x)}-(ax+b)\right|=0$$

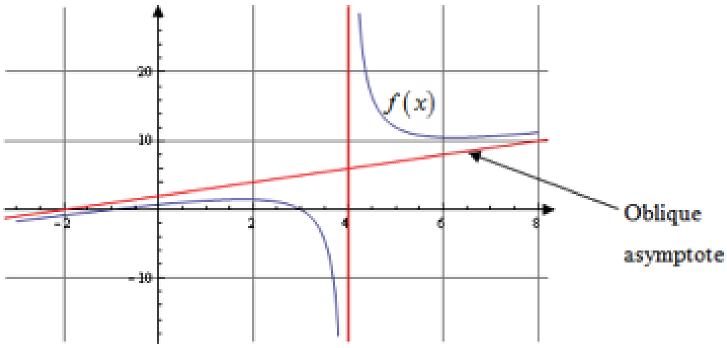
$$\lim_{x\to-\infty}\left|\frac{P(x)}{Q(x)}-(ax+b)\right|=0$$

These results tell us that the graph of the equation $y=\frac{P(x)}{Q(x)}$ tends towards the line (oblique asymptote) $y=ax+b$ as $x\to+\infty$ or $x\to-\infty$.

Another way to find the values of constants a and b is $a=\lim_{x\to\pm\infty}\frac{f(x)}{x}$ $a\neq 0$ and $b=\lim_{x\to\pm\infty}\left[f(x)-ax\right]$

Notice

Horizontal asymptote and oblique asymptote do not exist on the same side. That means if $f(x)\to L$ as $x\to+\infty$, there is no oblique asymptote on the right side since there is horizontal asymptote and if $f(x)\to L$ as $x\to-\infty$, there is no oblique asymptote on the left side since there is horizontal asymptote.



Example 61

Find the oblique asymptote of $f(x)=\frac{3x^3+4x-5}{x^2+1}$

Solution

Let $f(x)=\frac{3x^3+4x-5}{x^2+1}$

$Domf=]-\infty,+\infty[$

Let $y=ax+b$ be the oblique asymptote.

$$a=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{3x^3+4x-5}{x^3+x}$$

$$=3$$

$$b=\lim_{x\to\pm\infty}\left[f(x)-3x\right]$$

$$=\lim_{x\to\pm\infty}\frac{3x^3+4x-5-3x^3-3x}{x^2+1}$$

$$=\lim_{x\to\pm\infty}\frac{x-5}{x^2+1}$$

$$=0$$

Thus, $y=3x$ is the oblique asymptote for $f(x)=\frac{3x^3+4x-5}{x^2+1}$

Example 62

Let $f(x)=\frac{x}{x-2}$. Find relative asymptotes;

Solution

$Domf=(-\infty,2)\cup(2,+\infty)$

$\lim_{x\to2^+}f(x)=+\infty$ and $\lim_{x\to2^-}f(x)=-\infty$. Thus, there exists a vertical asymptote $\text{V.A}\equiv x=2$

$\lim_{x\to\pm\infty}f(x)=1$. Thus, there exists a horizontal asymptote

$\text{H.A}\equiv y=1$

Note that there is no oblique asymptote.

Example 63

Let $f(x)=\frac{x^2+2x-3}{x}$. Find relative asymptotes;

Solution

$Domf = (-\infty, 0) \cup (0, +\infty)$

$\lim_{x \to 0^+} f(x) = -\infty$ and $\lim_{x \to 0^-} f(x) = +\infty$. Thus, there is a vertical asymptote $V.A \equiv x = 0$

$\lim_{x \to \pm\infty} f(x) = \pm\infty$. Thus, the horizontal asymptote does not exist.

To find oblique asymptote, let $y = ax + b$ be the oblique asymptote.

$$a = \lim_{x \to \pm\infty} \frac{f(x)}{x}$$
$$= \lim_{x \to \pm\infty} \frac{x^2 + 2x - 3}{x^2}$$
$$= 1$$

Since $1 \neq 0$, let us find b.

$$b = \lim_{x \to \pm\infty} \left[f(x) - ax \right]$$
$$= \lim_{x \to \pm\infty} \left[\frac{x^2 + 2x - 3}{x} - x \right]$$
$$= \lim_{x \to \pm\infty} \frac{2x - 3}{x}$$
$$= 2$$

Then, $O.A \equiv y = x + 2$.

As the degree of the numerator exceeds the degree of the denominator by one, we could find oblique asymptote after performing long division.

x	$x + 2$
	$x^2 + 2x - 3$
	$-(x^2)$
	$2x - 3$
	$-(2x)$
	-3

$f(x) = x + 2 - \frac{3}{x}$.

Thus, there exists an oblique asymptote $O.A \equiv y = x + 2$.

Example 64

Let $f(x)=x+|x|+\frac{1}{x}$. Find relative asymptotes

Solution

$$Domf=\mathbb{R}\setminus\{0\}=]-\infty,0[\cup]0,+\infty[$$

$$f(x)=\begin{cases} x+x+\frac{1}{x}=2x+\frac{1}{x} & if\ x>0 \\ x-x+\frac{1}{x}=\frac{1}{x} & if\ x<0 \end{cases}$$

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0}\left(2x+\frac{1}{x}\right)=\infty$ and $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0}\left(\frac{1}{x}\right)=\infty$.

Thus, there is a vertical asymptote $V.A\equiv x=0$

$\lim\limits_{x\to +\infty} f(x)=\lim\limits_{x\to +\infty}\left(2x+\frac{1}{x}\right)=+\infty$ and $\lim\limits_{x\to -\infty} f(x)=\lim\limits_{x\to -\infty}\frac{1}{x}=0$.

Thus, there is a horizontal asymptote $H.A\equiv y=0$ on the left side.

As there is no horizontal asymptote on the right side, let us check if there is oblique asymptote.

Let $O.A\equiv y=ax+b$

$$\lim_{x\to +\infty}\frac{f(x)}{x}=\lim_{x\to +\infty}\left(2+\frac{1}{x^2}\right)=2$$

Then $a=2$

$$\lim_{x\to +\infty}\left[f(x)-ax\right]=\lim_{x\to +\infty}\left(2x+\frac{1}{x}-2x\right)=0.$$

Then $b=0$

Thus, on the right side there is oblique asymptote $O.A\equiv y=2x$

Example 65

Let $f(x)=2x-\sqrt{4x^2+1}$. Find relative asymptotes

Solution

$Domf = (-\infty, +\infty)$

From $Domf$, there is no vertical asymptotes.

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(2x-\sqrt{4x^2+1}\right)$$
$$= +\infty - \infty \quad I.F$$

Remove this I.F

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\frac{\left(2x-\sqrt{4x^2+1}\right)\left(2x+\sqrt{4x^2+1}\right)}{2x+\sqrt{4x^2+1}}$$
$$= \lim_{x\to+\infty}\frac{4x^2-\left(4x^2+1\right)}{2x+\sqrt{4x^2+1}}$$
$$= \lim_{x\to+\infty}\frac{-1}{2x+\sqrt{4x^2+1}}$$
$$= 0$$

Thus, on the right side, $y = 0$ is horizontal asymptote and hence no oblique asymptote on the right side.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(2x-\sqrt{4x^2+1}\right)$$
$$= -\infty - \infty$$
$$= -\infty$$

Thus, there is no horizontal asymptote on the left side.

Oblique asymptote;

Let $O.A \equiv y = ax + b$

$$a = \lim_{x\to-\infty}\frac{f(x)}{x} = \lim_{x\to-\infty}\frac{2x-\sqrt{4x^2+1}}{x}$$
$$= \lim_{x\to-\infty}\left(2-\frac{-x\sqrt{4+\frac{1}{x^2}}}{x}\right)$$
$$= 4$$

$$b = \lim_{x\to-\infty}\left[f(x)-4x\right]$$

$$= \lim_{x\to-\infty}\left(2x-\sqrt{4x^2+1}-4x\right)$$

$$= \lim_{x\to-\infty}\left(-2x-\sqrt{4x^2+1}\right)$$

$$= \lim_{x\to-\infty}\frac{\left(-2x-\sqrt{4x^2+1}\right)\left(-2x+\sqrt{4x^2+1}\right)}{\left(-2x+\sqrt{4x^2+1}\right)}$$

$$= \lim_{x\to-\infty}\frac{4x^2-4x^2-1}{\left(-2x+\sqrt{4x^2+1}\right)}$$

$$= \lim_{x\to-\infty}\frac{-1}{\left(-2x+\sqrt{4x^2+1}\right)}$$

$$=0$$

Thus, $y=4x$ is the oblique asymptote on the left side.

Example 66

Let $f(x)=\dfrac{x^2+3}{\sqrt{2x^2+x-1}}$. Find relative asymptotes

Solution

$$Domf = \left]-\infty,-1\right[\cup\left]\frac{1}{2},+\infty\right[$$

$$\lim_{x\to-1}f(x)=\lim_{x\to-1}\frac{x^2+3}{\sqrt{2x^2+x-1}}$$
$$=\infty$$

$x=-1$ is a vertical asymptote

$$\lim_{x\to\frac{1}{2}}f(x)=\lim_{x\to\frac{1}{2}}\frac{x^2+3}{\sqrt{2x^2+x-1}}$$
$$=\infty$$

$x=\dfrac{1}{2}$ is another vertical asymptote for $f(x)=\dfrac{x^2+3}{\sqrt{2x^2+x-1}}$

$$\lim_{x\to+\infty}f(x)=\lim_{x\to+\infty}\frac{x^2+3}{\sqrt{2x^2+x-1}}$$

$$=\frac{+\infty}{+\infty}\quad I.F$$

Remove this I.F

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2\left(1+\frac{3}{x^2}\right)}{x\sqrt{2+\frac{1}{x}-\frac{1}{x^2}}}$$

$$= \lim_{x\to+\infty} \frac{x\left(1+\frac{3}{x^2}\right)}{\sqrt{2+\frac{1}{x}-\frac{1}{x^2}}}$$

$$= +\infty$$

Thus, no horizontal asymptote when $x \to +\infty$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2+3}{\sqrt{2x^2+x-1}}$$

$$= \frac{+\infty}{+\infty} \quad I.F$$

Remove this $I.F$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2\left(1+\frac{3}{x^2}\right)}{-x\sqrt{2+\frac{1}{x}-\frac{1}{x^2}}}$$

$$= \lim_{x\to-\infty} \frac{x\left(1+\frac{3}{x^2}\right)}{-\sqrt{2+\frac{1}{x}-\frac{1}{x^2}}}$$

$$= +\infty$$

Thus, no horizontal asymptote when $x \to -\infty$.

Oblique asymptote:

Let $O.A \equiv y = ax + b$

$$a = \lim_{x\to+\infty} \frac{f(x)}{x}$$

$$= \lim_{x\to+\infty} \frac{x^2+3}{x\sqrt{2x^2+x-1}}$$

$$= \frac{1}{\sqrt{2}}$$

$$b=\lim_{x\to+\infty}\left[f(x)-\frac{1}{\sqrt{2}}x\right]$$

$$=\lim_{x\to+\infty}\left[\frac{x^2+3}{\sqrt{2x^2+x-1}}-\frac{x}{\sqrt{2}}\right]$$

$$=\lim_{x\to+\infty}\left(\frac{\sqrt{2}\left(x^2+3\right)-x\sqrt{2x^2+x-1}}{\sqrt{4x^2+2x-2}}\right)$$

$$=\lim_{x\to+\infty}\left(\frac{\sqrt{2}\left(x^2+3\right)-x\sqrt{2x^2+x-1}}{\sqrt{4x^2+2x-2}}\right)\left(\frac{\sqrt{2}\left(x^2+3\right)+x\sqrt{2x^2+x-1}}{\sqrt{4x^2+2x-2}}\right)\left(\frac{\sqrt{4x^2+2x-2}}{\sqrt{2}\left(x^2+3\right)+x\sqrt{2x^2+x-1}}\right)$$

$$=\lim_{x\to+\infty}\left(\frac{2\left(x^2+3\right)^2-x^2\left(2x^2+x-1\right)}{4x^2+2x-2}\right)\left(\frac{\sqrt{4x^2+2x-2}}{\sqrt{2}\left(x^2+3\right)+x\sqrt{2x^2+x-1}}\right)$$

$$=\lim_{x\to+\infty}\left(\frac{2x^4+12x^2+18-2x^4-x^3+x^2}{4x^2+2x-2}\right)\left(\frac{\sqrt{4x^2+2x-2}}{\sqrt{2}\left(x^2+3\right)+x\sqrt{2x^2+x-1}}\right)$$

$$=\lim_{x\to+\infty}\left(\frac{13x^2+18-x^3}{4x^2+2x-2}\right)\left(\frac{\sqrt{4x^2+2x-2}}{\sqrt{2}\left(x^2+3\right)+x\sqrt{2x^2+x-1}}\right)$$

$$=\lim_{x\to+\infty}\left[\left(\frac{13x^2+18-x^3}{4x^2+2x-2}\right)\left(\frac{x\sqrt{4+\frac{2}{x}-\frac{2}{x^2}}}{x^2\left(\sqrt{2}+\frac{3\sqrt{2}}{x^2}+\sqrt{2+\frac{1}{x}-\frac{1}{x^2}}\right)}\right)\right]$$

$$=\lim_{x\to+\infty}\left[\left(\frac{13x^2+18-x^3}{4x^2+2x-2}\right)\left(\frac{\sqrt{4+\frac{2}{x}-\frac{2}{x^2}}}{x\left(\sqrt{2}+\frac{3\sqrt{2}}{x^2}+\sqrt{2+\frac{1}{x}-\frac{1}{x^2}}\right)}\right)\right]$$

$$= \lim_{x\to\circ}\left(\left(\frac{13x^2+18-x^3}{4x^3+2x^2-2x}\right)\frac{\sqrt{4+\frac{2}{x}-\frac{2}{x^2}}}{\sqrt{2+\frac{3\sqrt{2}}{x^2}}+\sqrt{2+\frac{1}{x}-\frac{1}{x^2}}}\right)$$

$$= \lim_{x\to\circ}\frac{x3\left(\frac{13}{x}+\frac{18}{x^3}-1\right)}{x^3\left(4+\frac{2}{x}-\frac{2}{x^2}\right)}\frac{\sqrt{4+\frac{2}{x}-\frac{2}{x^2}}}{\sqrt{2+\frac{3\sqrt{2}}{x^2}}+\sqrt{2+\frac{1}{x}-\frac{1}{x^2}}}$$

$$= \lim_{x\to\circ}\frac{\left(\frac{13}{x}+\frac{18}{x^3}-1\right)}{\left(4+\frac{2}{x}-\frac{2}{x^2}\right)}\frac{\sqrt{4+\frac{2}{x}-\frac{2}{x^2}}}{\left(\sqrt{2+\frac{3\sqrt{2}}{x^2}}+\sqrt{2+\frac{1}{x}-\frac{1}{x^2}}\right)}$$

$$= \frac{-1}{4\sqrt{2}}$$

$$= \frac{-\sqrt{2}}{8}$$

Hence, $y=\frac{\sqrt{2}x}{2}-\frac{\sqrt{2}}{8}$ is oblique asymptote when $x\to+\infty$.

Let us check if there is oblique asymptote when $x\to-\infty$.

$$a = \lim_{x\to-\infty}\frac{f(x)}{x}$$

$$= \lim_{x\to-\infty}\frac{x^2+3}{x\sqrt{2x^2+x-1}}$$

$$= -\frac{1}{\sqrt{2}}$$

$$b=\lim_{x\to-\infty}\left[f(x)+\frac{1}{\sqrt{2}}x\right]$$

$$=\lim_{x\to-\infty}\left[\frac{x^2+3}{\sqrt{2x^2+x-1}}+\frac{x}{\sqrt{2}}\right]$$

$$=\lim_{x\to-\infty}\left(\frac{\sqrt{2}\left(x^2+3\right)+x\sqrt{2x^2+x-1}}{\sqrt{4x^2+2x-2}}\right)$$

$$=\lim_{x\to-\infty}\left(\frac{\sqrt{2}\left(x^2+3\right)+x\sqrt{2x^2+x-1}}{\sqrt{4x^2+2x-2}}\right)\left(\frac{\sqrt{2}\left(x^2+3\right)-x\sqrt{2x^2+x-1}}{\sqrt{4x^2+2x-2}}\right)\left(\frac{\sqrt{4x^2+2x-2}}{\sqrt{2}\left(x^2+3\right)-x\sqrt{2x^2+x-1}}\right)$$

$$=\lim_{x\to-\infty}\left(\frac{2\left(x^2+3\right)^2-x^2\left(2x^2+x-1\right)}{4x^2+2x-2}\right)\left(\frac{\sqrt{4x^2+2x-2}}{\sqrt{2}\left(x^2+3\right)-x\sqrt{2x^2+x-1}}\right)$$

$$=\lim_{x\to-\infty}\left(\frac{2x^4+12x^2+18-2x^4-x^3+x^2}{4x^2+2x-2}\right)\left(\frac{\sqrt{4x^2+2x-2}}{\sqrt{2}\left(x^2+3\right)-x\sqrt{2x^2+x-1}}\right)$$

$$=\lim_{x\to-\infty}\left(\frac{13x^2+18-x^3}{4x^2+2x-2}\right)\left(\frac{\sqrt{4x^2+2x-2}}{\sqrt{2}\left(x^2+3\right)-x\sqrt{2x^2+x-1}}\right)$$

$$=\lim_{x\to-\infty}\left[\left(\frac{13x^2+18-x^3}{4x^2+2x-2}\right)\left(\frac{-x\sqrt{4+\frac{2}{x}-\frac{2}{x^2}}}{x^2\left(\sqrt{2}+\frac{3\sqrt{2}}{x^2}+\sqrt{2+\frac{1}{x}-\frac{1}{x^2}}\right)}\right)\right]$$

$$=\lim_{x\to-\infty}\left[\left(\frac{13x^2+18-x^3}{4x^2+2x-2}\right)\left(\frac{-\sqrt{4+\frac{2}{x}-\frac{2}{x^2}}}{x\left(\sqrt{2}+\frac{3\sqrt{2}}{x^2}+\sqrt{2+\frac{1}{x}-\frac{1}{x^2}}\right)}\right)\right]$$

$$= \lim_{x\to\circ}\left(\left(\frac{13x^2+18-x^3}{4x^3+2x^2-2x}\right)\frac{-\sqrt{4+\frac{2}{x}-\frac{2}{x^2}}}{\sqrt{2+\frac{3\sqrt{2}}{x^2}}+\sqrt{2+\frac{1}{x}-\frac{1}{x^2}}}\right)$$

$$= \lim_{x\to\circ}\frac{x^3\left(\frac{13}{x}+\frac{18}{x^3}-1\right)}{x^3\left(4+\frac{2}{x}-\frac{2}{x^2}\right)}\frac{-\sqrt{4+\frac{2}{x}-\frac{2}{x^2}}}{\sqrt{2+\frac{3\sqrt{2}}{x^2}}+\sqrt{2+\frac{1}{x}-\frac{1}{x^2}}}$$

$$= \lim_{x\to\circ}\frac{\left(\frac{13}{x}+\frac{18}{x^3}-1\right)}{\left(4+\frac{2}{x}-\frac{2}{x^2}\right)}\left(\frac{-\sqrt{4+\frac{2}{x}-\frac{2}{x^2}}}{\sqrt{2+\frac{3\sqrt{2}}{x^2}}+\sqrt{2+\frac{1}{x}-\frac{1}{x^2}}}\right)$$

$$= \frac{1}{4\sqrt{2}}$$

$$= \frac{\sqrt{2}}{8}$$

Hence, $y=-\frac{\sqrt{2}x}{2}+\frac{\sqrt{2}}{8}$ is oblique asymptote when $x\to-\infty$.

Exercise 12

Find relative asymptotes of;

1. $f(x)=\frac{x^3+x^2-5x-2}{x^3-x^2-2x}$
2. $y=\frac{x+3}{x^2+9}$
3. $y=\frac{x^2+3x+1}{4x-9}$
4. $y=\frac{x^2-x-2}{x-2}$
5. $f(x)=\frac{6x^2-3x+4}{2x^2-8}$

Link to other subjects

In physics,

Instantaneous velocity:

Instantaneous velocity is the limit of average velocity over an infinitesimal interval of time.

$$v = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t}$$

Instantaneous acceleration:

Instantaneous acceleration is the limit of average acceleration over an infinitesimal interval of time.

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

(Here and elsewhere, if motion is in a straight line, vector quantities can be substituted by scalars in the equations.)

Unit summary

1. A set N is called a neighbourhood of point p if there exist an open interval I such that $x \in I \subset N$. The collection of all neighbourhoods of a point is called the **neighbourhood system** at the point. A **deleted neighbourhood** of a point p (sometimes called a **punctured neighbourhood**) is a neighbourhood of p without p itself.

2. To find limit of a function $f(x)$ as x approaches a, first we need to substitute that value a in the function and see what happen. The limit can exist or not.

3. If the value of $f(x)$ approaches L_1 as x approaches x_0 from the right side we write $\lim_{x \to x_0^+} f(x) = L_1$ and we read "**the limit of $f(x)$ as x approaches x_0 from the right equals L_1**.

4. If the value of $f(x)$ approaches L_2 as x approaches x_0 from the left side we write $\lim\limits_{x\to x_0^-} f(x)=L_2$ and we read "**the limit of** $f(x)$ **as** x **approaches** x_0 **from the left equals** L_2

5. Squeeze theorem: Suppose that $f(x)<h(x)<g(x)$. If $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c} g(x)=L$, then $\lim\limits_{x\to c} h(x)=L$

6. Operations on limits

   Let lim stands for one of the limits $\lim\limits_{x\to a}$, $\lim\limits_{x\to a^-}$, $\lim\limits_{x\to a^+}$, $\lim\limits_{x\to -\infty}$ or $\lim\limits_{x\to +\infty}$. If $\lim f(x)$ and $\lim g(x)$ both exist, say $\lim f(x)=L_1$ and $\lim g(x)=L_2$, then

   - A constant factor can be moved through a limit sign. That is, if k is a constant, then $\lim\left[kf(x)\right]=k\lim f(x)$ $=\mathrm{kL}_1$
   - $\lim\left[f(x)+g(x)\right]=\lim f(x)+\lim g(x)=L_1+L_2$
   - $\lim\left[f(x)-g(x)\right]=\lim f(x)-\lim g(x)=L_1-L_2$
   - $\lim\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{\lim f(x)}{\lim g(x)}=\dfrac{L_1}{L_2}$ if $L_2\neq 0$
   - If n and m are positive integers, then $\lim\left[f(x)\right]^{\frac{m}{n}}=L_1^{\frac{m}{n}}$ provided that $L_1\geq 0$ if n is even.

7. The types $\dfrac{0}{0},\dfrac{\infty}{\infty},0\cdot\infty,\infty-\infty$ are indeterminate forms.

8. When we are computing the limits of irrational functions, in case of indeterminate form, we need to know the conjugate of the irrational expression in that function. We may need to find the domain of the given function.

9. A function $f(x)$ is said to be **continuous at point *c*** if the following conditions are satisfied:

   a) $f(c)$ is defined
   b) $\lim_{x\to c} f(x)$ exists
   c) $\lim_{x\to c} f(x) = f(c)$

10. If a function is undefined or discontinuous at a point a but can be redefined at that single point so that it becomes continuous there, then we say that f has a **removable** (or **apparent**) discontinuity at a.

11. If $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ exist but are different, $f(x)$ is said to have a discontinuity of the first kind at point c and c is called discontinuity point of the first kind.

12. A function f is said to have a discontinuity of the second kind at point c if at least one of its limits from the left or from the right does not exist or is infinity.

13. A line $x = x_0$ is called a **vertical asymptote** for the graph of a function $f(x)$ if $f(x)\to+\infty$ or $f(x)\to-\infty$ as x approaches x_0 at the right or at the left.

14. A line $y = L$ is called a **horizontal asymptote** for the graph of a function $f(x)$ if $f(x)\to L$ as $x\to+\infty$ or $x\to-\infty$.

15. Oblique asymptote: $y = ax + b$ as $x\to+\infty$ or $x\to-\infty$. Where

$$a = \lim_{x\to\pm\infty}\frac{f(x)}{x} \quad a\neq 0 \text{ and } b = \lim_{x\to\pm\infty}\left[f(x) - ax\right]$$

Revision exercise

From exercise 1 to 10, evaluate the given limits

1. $\lim\limits_{x\to 2}\frac{x^2-3x+2}{x^2-6x+8}$ 2. $\lim\limits_{x\to 1}\frac{\sqrt{5x-4}-\sqrt{x}}{x-1}$ 3. $\lim\limits_{x\to 1}\frac{x-1}{\sqrt{x^2+3}-2}$

4. $\lim\limits_{n\to\infty}\left(\sqrt{n^2+n}-n\right)$ 5. $\lim\limits_{x\to -1}\left(x^5-3x^2-3x+9\right)$ 6. $\lim\limits_{x\to 1}\frac{x^6-2x^2+1}{x^2-1}$

7. $\lim\limits_{x\to 2} f(x)$ for $f(x)=\begin{cases} x^2 & \text{if } 0\le x\le 2 \\ x+2 & \text{if } 2<x\le 6\end{cases}$

8. $\lim\limits_{x\to 3} f(x)$ for $f(x)=\begin{cases} x-1 & x\le 3 \\ 3x-7 & x>3\end{cases}$

9. $\lim\limits_{t\to 0} g(t)$ for $g(t)=\begin{cases} t^2 & t\ge 0 \\ t-2 & t<0\end{cases}$

10. Let the function $f(x)$ be defined by

$$f(x)=\begin{cases} \frac{x^2-9}{x+3} & x\neq -3 \\ k & x=-3\end{cases}$$

Determine k if $f(-3)=\lim\limits_{x\to -3} f(x)$.

11. A function $\phi(x)$ is defined as $\phi(x)=\begin{cases} 1+x, & x<2 \\ 5-x, & x\ge 2\end{cases}$. Is this function continuous at $x=2$?

12. For the function $f(x)=\begin{cases} -x, & x<0 \\ a, & x=0 \\ x^2, & x>0\end{cases}$ find the value of a that will make it continuous at $x=0$.

13. A function is defined as $f(x)=\begin{cases} x, & 0\le x<1 \\ 2, & x=1 \\ x+1, & 1<x\le 2\end{cases}$. Find the point(s) of discontinuity of the function $f(x)$ and draw its graph.

14. Let $f(x)$ be a function of real variable x defined by

$$f(x)=\begin{cases} -x, & x\le 0 \\ x, & 0<x<1 \\ 2-x, & x\ge 1\end{cases}.$$

Show that $f(x)$ is continuous at $x=0$ and also at $x=1$

In Exercises 15-17, classify the points of discontinuity (if any):

15. $f(x)=\frac{x^2-16}{x-4}$ 16. $f(x)=\frac{x^2+2x+5}{x+2}$ 17. $f(x)=\begin{cases}x-1 & x\le 3\\ 2x-7 & x>3\end{cases}$

Find relative asymptotes in exercises 18-31:

18. $y=\sqrt{\frac{a+x}{a-x}},a>0$ 19. $y=\sqrt{\frac{x\left(x^2-a^2\right)}{a}},a>0$ 20. $y=\sqrt{\frac{x^3}{x-1}}$

21. $y=\frac{x^2}{a}+\frac{a^2}{x},a>0$ 22. $y=\frac{2x^2-2x-1}{x-2}$ 23. $y=x+\sqrt{\frac{x-1}{x+1}}$

24. $y=x\sqrt{\frac{x-1}{x-2}}$ 25. $y=\frac{2-2x}{5-3x}$ 26. $y=\frac{x^4}{x^2+1}$

27. $y=ax+\frac{1}{ax},a>0$ 28. $y=\frac{2x^2-1}{x^2-1}$ 29. $y=\frac{x^3}{x-a},a>0$

30. $y=\frac{x^3-2x+1}{2x^2+2x+2}$ 31. $y=x\sqrt{\frac{x^2-1}{2x-1}}$

32. Find horizontal asymptote(s): $y=\frac{x^3-2}{|x|^3+1}$

33. Find horizontal and vertical asymptotes of $f(x)=\frac{1}{\sqrt{x^2-2x}-x}$

Unit 6

Differentiation of polynomial, rational and irrational functions

My goals

By the end of this unit, I will explain:

- Concepts of derivative of a function.
- Rules of differentiation.
- Applications.

Introduction

Calculus is concerned with things that do not change at a constant rate. The values of the function called the derivative will be that varying rate of change.

1. It is used economics a lot, calculus is also a base of economics.
2. It is used in history, for predicting the life of a stone.
3. It is used in geography, which is used to study the gases present in the atmosphere.
4. It is mainly used in daily life by pilots to measure the pressure in the air.

Differentiation can help us solve many types of real-world problems. We use the derivative to determine the maximum and minimum values of particular functions (e.g. Cost, strength, amount of material used in a building, profit, loss, etc.). Derivatives are met in many engineering and science problems, especially when modeling the behavior of moving objects.

Required outcomes

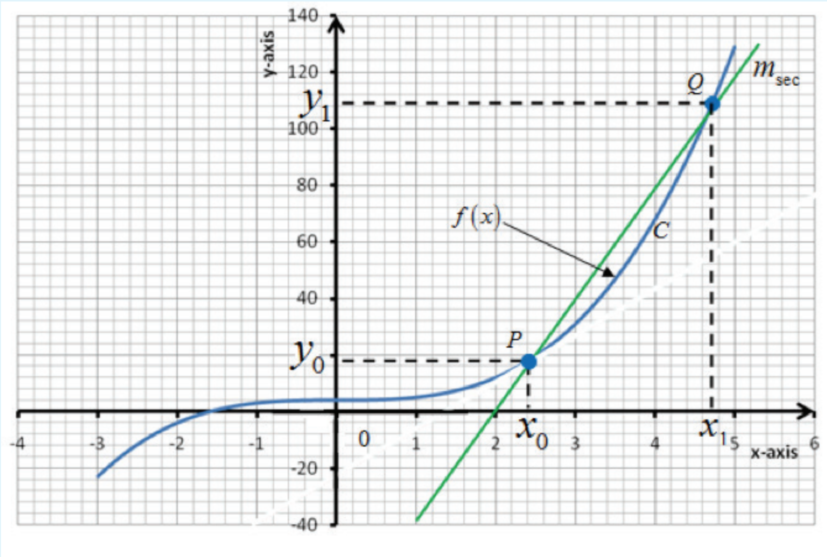
After completing this unit, the learners should be able to:

- Use properties of derivatives to differentiate polynomial, rational ad irrational functions
- Use first principles to determine the gradient of the tangent line to a curve at a point.
- Apply the concepts of and techniques of differentiation to model, analyze and solve rates or optimization problems in different situations.
- Use the derivative to find the equation of a line tangent or normal to a curve at a given point.

1. Concepts of derivative of a function

Activity 1

Consider the following figure



1. If $P(x_0, y_0)$ and $Q(x_1, y_1)$ are two points on the graph of a function f, find the slope of secant line (m_{sec}) passing through P and Q.

Since $y_0 = f(x_0)$ and $y_1 = f(x_1)$, express in terms of $f(x_0)$ and $f(x_1)$.

2. If we let x_1 approach x_0, how can you conclude about position of Q to P ?
3. Let $m_{\tan} = \lim_{x_1 \to x_0} m_{\sec}$, write down expression of $m_{\tan}$ in terms of $f(x_0)$ and $f(x_1)$.
4. After letting $h = x_1 - x_0$, rewrite $m_{\tan}$ in terms of $f(x_0)$ and $f(x_0 + h)$

If $P(x_0, y_0)$ is a point on the graph of a function f,

$m_{\tan} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$ is called **slope of tangent line** to the graph of f at P; if this limit exists.

$m_{\tan}$ has a special notation, we denote it by

$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$ and $f'(x_0)$ is read f prime of x_0

Dropping the subscript on x_0 in notation of $m_{\tan}$, we get one of the most important in mathematics, the derivative of a function.

The derivative of a function $f(x)$ with respect to x is

denoted by $f'(x)$ and defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided that the limit exists.

Example 1

Let $f(x)=x^2+1$, find $f'(x)$

Solution

The derivative of $f(x)$ is

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{(x+h)^2+1-x^2-1}{h}$$
$$=\lim_{h\to 0}\frac{x^2+2hx+h^2+1-x^2-1}{h}$$
$$=\lim_{h\to 0}\frac{2hx+h^2}{h}$$
$$=\lim_{h\to 0}(2x+h)$$
$$=2x$$

Thus, $f'(x)=2x$

Example 2

Let $f(x)=2x^2+3$, find $f(4)$

Solution

The derivative of $f(x)$ at $x=4$ is

$$f'(4)=\lim_{x\to 4}\frac{f(x)-f(4)}{x-4}$$
$$=\lim_{x\to 4}\frac{2x^2+3-35}{x-4}$$
$$=\lim_{x\to 4}\frac{2x^2-32}{x-4}$$
$$=\lim_{x\to 4}\frac{2(x+4)(x-4)}{x-4}$$
$$=\lim_{x\to 4}2(x+4)$$
$$=16$$

Thus, $f'(4)=16$

Remarks:

- If $t\to f(t)$ represents the law of a moving object, then the derivative number of f represents the instantaneous speed of that moving object at instant t.
- The process of finding derivative of a function is called **differentiation** of that function.

Exercise 1

Find the derivative of

1. $f(x)=x+3$ at $x=1$
2. $f(x)=x^3+3$ at $x=-2$
3. $f(x)=4x^2-x+3$
4. $f(x)=4x^2+3x-4$
5. $f(x)=4$

Right-hand and left-hand derivatives

Activity 2

1. Consider the function

$$f(x)=\begin{cases} x+2, & x>2 \\ 4, & x\le 2 \end{cases}$$

Find $f'(x)$ from the left of 2 and $f'(x)$ from the right of 2

2. Consider the function

$$f(x)=\begin{cases} 3x-2, & x\le 3 \\ x+4 & x>3 \end{cases}$$

Find $f'(x)$ from the left of 3 and $f'(x)$ from the right of 3.

Let $y=f(x)$ be a function and let x_0 be in the domain of f.

The right- hand derivative of f at $x=x_0$ is the number

$$f'(x_0^+)=\lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}$$

And the left- hand derivative is the number

$$f'(x_0^-)=\lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}.$$

The function $f(x)$ is said to be **differentiable** at $x=x_0$ if and only if $f(x)$ has both a right- hand and a left- hand derivatives and all of them are equal.

The function $f(x)$ is said to be **differentiable** on interval I if it is differentiable at every point of that interval.

Example 3

Let $f(x)=|x|$

$$f(x)=\begin{cases} x & \text{if } x\ge 0 \\ -x & \text{if } x<0 \end{cases}$$

$$f'(0^+) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$$
$$= \lim_{x\to 0^+} \frac{x-0}{x-0}$$
$$=1$$
$$f'(0^-) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$$
$$= \lim_{x\to 0^-} \frac{-x-0}{x-0}$$
$$=-1$$

As $f'(0^+) \neq f'(0^-)$, then $f(x)=|x|$ is not differentiable at 0.

Exercise 2

1. Consider the function $f(x)=\begin{cases} x+2, & x>1 \\ 2x-3 & x\le 1\end{cases}$. Find $f'(x)$ at $x=1$
2. Consider the function $f(x)=\begin{cases} x^2-4, & x>2 \\ x-2 & x\le 2\end{cases}$. Find $f'(x)$ at $x=2$
3. Consider the function $f(x)=\frac{|x|}{4}$. Find $f'(x)$ at $x=0$
4. Find $f'(4)$ if $f(x)=\begin{cases} 2x-4, & x\le 4 \\ x & x>4\end{cases}$
5. Find $f'(-1)$ if $f(x)=\frac{|x+1|}{x+1}$

Notation

We have used the notation $f'(x)$ to denote the derivative of the function $f(x)$. There are **many other ways to denote the derivative** of a function:

$\frac{df}{dx}$ and $D_x f$ are used by some authors to denote the derivative of the function $f(x)$.

If we consider $y = f(x)$, then y' denotes the derivative of the function $f(x)$.

If the function $f(x)$ is differentiable on the interval I then $f(x)$ is continuous on I.

Note that the converse of this theorem is not true.

Example 4

From example 3, we have seen that $f(x) = |x|$ is not differentiable at 0. But this function is continuous at 0.

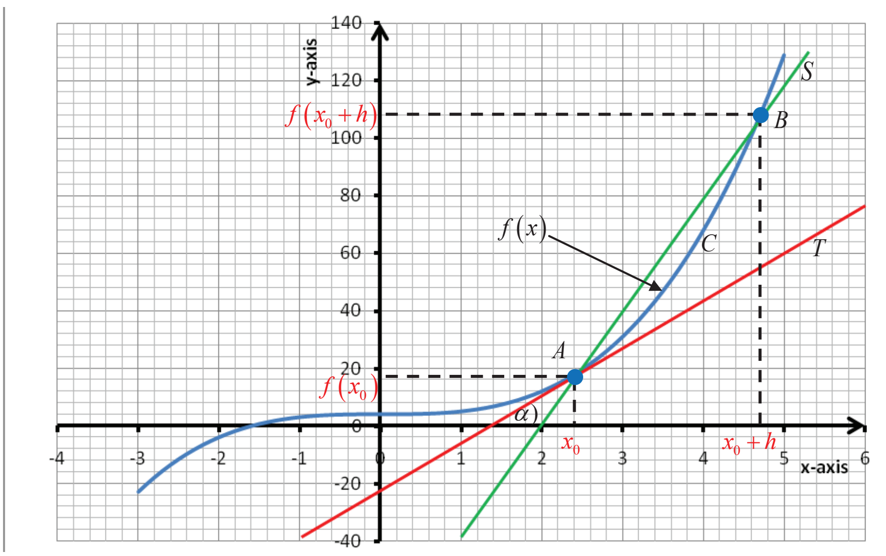
$$f(x) = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} x = 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} (-x) = 0$$

As $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0$, then $\lim_{x \to 0} f(x) = 0 = f(0)$ and hence $f(x)$ is continuous at 0.

Geometric interpretation of derivative



Let *A* and *B* be two points of the curve *C* of the function $f(x)$

AB has slope $\frac{f(x_0+h)-f(x_0)}{(x_0+h)-x_0}=\frac{f(x_0+h)-f(x_0)}{h}$.

When *h* approaches zero, the point *B* approaches point *A*. At this time, the secant line *S* will approach the tangent line *T* to the curve *C* at point *A*.

Then the slope of tangent line is $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=\tan\alpha$

The slope of the tangent line to the curve at a point is equal to the derivative of the function at that point:

$\tan\alpha = f'(x)=\lim_{\Delta x\to 0}\frac{\Delta f(x)}{\Delta x}$

is the angle between *x-axis* and the tangent line of $y=f(x)$ at point x_0.

$f'(x_0)$ is the slope of tangent line of $y=f(x)$ at point $(x_0, f(x_0))$.

Later we shall see how to find the equation of the tangent line.

2. Rules of differentiation

a) Constant function and Powers

Activity 3

Let $D(I,\mathbb{R})$ be the set of functions differentiable on I.

1. If *f* is a constant function, say $f(x)=c$, for all x, use definition of derivative to find the derivative $f^{'}(x)$.
2. If *f* is a monomial function, with coefficient 1, say $f(x)=x^n$ for all real number n, use definition of derivative to find the derivative $f^{'}(x)$.

Derivative of a constant function

From activity 1

If f is a constant function, say $f(x)=c$, for all x, then

$$\frac{df}{dx}=\frac{d}{dx}(c)=0$$

Derivative of a power

If $\boldsymbol{n}$ is any real number, then

$\frac{d}{dx}x^n=nx^{n-1}$ for all x where the powers x^n and x^{n-1} are defined.

for all x where the powers x^n and x^{n-1} are defined.

This holds for any function with power. Thus, if $f\in(D,I)$ for positive and negative, and fractional value of n,

$$\left(f^n\right)'=nf^{n-1}f'.$$

Particular case

Let $f(x)=\sqrt{g(x)}$.

Here $n=\frac{1}{2}$ because $\sqrt{g(x)}=[g(x)]^{\frac{1}{2}}$

The derivative of $f(x)$ is as follows

$$f'(x)=\frac{1}{2}[g(x)]^{\frac{1}{2}-1}g'(x)$$

$$=\frac{1}{2}[g(x)]^{-\frac{1}{2}}g'(x)$$

$$=\frac{1}{2}\frac{g'(x)}{[g(x)]^{\frac{1}{2}}}$$

$$=\frac{g'(x)}{2\sqrt{g(x)}}$$

Thus, if $f(x)=\sqrt{g(x)}$ then $f'(x)=\frac{g'(x)}{2\sqrt{g(x)}}$

Example 5

Find the derivative of $f(x)=123$.

Solution

This function is a constant function. Thus, $f'(x)=0$

Example 6

Differentiate the following powers of x

a) x^4 b) $\frac{1}{x^3}$

c) $x^{\frac{1}{2}}$ d) $x^{-\frac{3}{4}}$

e) $\sqrt{x^{2-\pi}}$

Solution

$$a)\ \frac{d}{dx}\left(x^4\right)=4x^{4-1}=4x^3$$

$$b)\ \frac{d}{dx}\left(\frac{1}{x^3}\right)=\frac{d}{dx}\left(x^{-3}\right)=-3x^{-3-1}=-3x^{-4}$$

$$c)\ \frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}x^{\frac{1}{2}-1}=\frac{1}{2}x^{-\frac{1}{2}}$$

$$d)\ \frac{d}{dx}\left(x^{\frac{-3}{4}}\right)=\frac{\frac{-3x^{-1}}{4}}{\frac{-3}{4}}$$

$$e)\ \frac{d}{dx}\left(\sqrt{x^{2-\pi}}\right)=\frac{d}{dx}\left(x^{\frac{2-\pi}{2}}\right)=\frac{2-\pi}{2}x^{\frac{2-\pi}{2}-1}$$

$$=\frac{2-\pi}{2}x^{-\frac{\pi}{2}}=\frac{2-\pi}{2}\sqrt{x^{-\pi}}$$

Example 7

Let $g(x)=(2x+1)^4$. *Find the derivative of* $g(x)$

Solution

$$g'(x)=4(2x+1)^3$$

$$=4\left(8x^3+12x^2+6x+1\right)(2)$$

$$=64x^3+96x^2+48x+8$$

Example 8

Let $f(x)=\sqrt{x^2+2}$.
Find the derivative of $f(x)$

Solution

$$f'(x)=\frac{\left(x^2+2\right)'}{2\sqrt{x^2+2}}=\frac{2x+0}{2\sqrt{x^2+2}}=\frac{x}{\sqrt{x^2+2}}$$

Exercise 3

Find the derivative of the following functions

1. $f(x)=67$
2. $g(x)=(x+1)^3$
3. $h(x)=\sqrt{2x^2+x-2}$

b) Multiplication by a scalar and product of two functions

Activity 4

Let $D(I,\mathbb{R})$ be the set of functions differentiable on I.

1. If *f* is a differentiable function of *x*, and c is a constant, use definition of derivative to find the derivative of $c\left[f(x)\right]$
2. If $f\in D(I,\mathbb{R})$ and $g\in D(I,\mathbb{R})$, use definition of derivative to find the derivative of the product $f(x)g(x)$

Multiplication by a scalar

From activity 2

If *f* is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}\left(cf(x)\right)=c\frac{d}{dx}f(x)$$

Derivative of a product

From activity 2:

If f *and* g are differentiable at x, then their product is $f \cdot g$, hence

$$\frac{d}{dx}(f \cdot g) = g\frac{df}{dx} + f\frac{dg}{dx}$$

Example 9

Find the derivative of

$f(x) = \frac{3}{2}\sqrt[3]{x}$

Solution

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(\frac{3}{2}\sqrt[3]{x}\right)$$

$$= \frac{3}{2}\frac{d}{dx}\left(x^{\frac{1}{3}}\right) = \frac{3}{2} \cdot \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{2}\sqrt[3]{x^{-2}}$$

Example 10

Find the derivative of

$f(x) = \frac{3}{2}\sqrt[3]{x}$

Example 10

Find the derivative of

$f(x) = \frac{3}{2}\sqrt[3]{x}$

Example 11

Let $f(x)=x\sqrt{x}$, find the derivative of $f(x)$

Solution

The derivative of $f(x)$ is denoted by $f'(x)$, then

$$f'(x)=x'\sqrt{x}+x\left(\sqrt{x}\right)'$$
$$=\sqrt{x}+x\frac{x'}{2\sqrt{x}}$$
$$=\sqrt{x}+\frac{x}{2\sqrt{x}}$$
$$=\sqrt{x}+\frac{\sqrt{x}}{2}$$
$$=\frac{3}{2}\sqrt{x}$$

Thus the derivative of $f(x)=x\sqrt{x}$ is $\frac{3}{2}\sqrt{x}$.

Example 12

Find the derivative of $y=\left(x^2+1\right)\left(x^3+3\right)$

Solution

$$y'=\left(x^2+1\right)'\left(x^3+3\right)+\left(x^2+1\right)\left(x^3+3\right)'$$
$$=(2x)\left(x^3+3\right)+\left(x^2+1\right)\left(3x^2\right)$$
$$=2x^4+6x+3x^4+3x^2$$
$$=5x^4+3x^2+6x$$

Example 13

Find the derivative of $y=\left(3-x^2\right)\left(x^3-x+1\right)$

Solution

From the product rule with $f(x)=3-x^2$ and $g(x)=x^3-x+1$, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[\left(3-x^2\right)\left(x^3-x+1\right)\right]$$
$$=\left(x^3-x+1\right)\frac{d}{dx}\left(3-x^2\right)+\left(3-x^2\right)\frac{d}{dx}\left(x^3-x+1\right)$$
$$=\left(x^3-x+1\right)(-2x)+\left(3-x^2\right)\left(3x^2-1\right)$$
$$=-2x^4+2x^2-2x+9x^2-3-3x^4+x^2$$
$$=-5x^4+12x^2-2x-3$$

Exercise 4

Find the derivative of the following functions

1. $f(x)=(x^2+6)(x-2)$
2. $g(x)=(x-3)(4x-5)$
3. $h(x)=5x^2(x-2)$
4. $k(x)=6(x-3)$

c) Sum (difference)of functions

Activity 5

Let $D(I,\mathbb{R})$ be the set of functions differentiable on I. If $f\in D(I,\mathbb{R})$ and $g\in D(I,\mathbb{R})$, use definition of derivative to find the derivative of the sum $f(x)+g(x)$ and the difference $f(x)-g(x)$

From activity 3

Let $D(I,\mathbb{R})$ be the set of functions differentiable on I.

If $f\in D(I,\mathbb{R})$ and $g\in D(I,\mathbb{R})$, then $f\pm g\in D(I,\mathbb{R})$.

In addition $\frac{d}{dx}(f\pm g)=\frac{df}{dx}\pm\frac{dg}{dx}$

Example 14

Find the derivative of $y=x^2-3x+7$

Solution

Differentiating $y=x^2-3x+7$ yields

$y'=(x^2)'-(3x)'+(7)'=2x-3$

Thus the derivative of $y=x^2-3x+7$ is $2x-3$.

Exercise 5

Find the derivative of the following functions

1. $f(x)=-4x^2+7x+5$
2. $g(x)=125x^6-215x^6+75$
3. $h(x)=24x^4-2x^3-85$

d) Reciprocal function and quotient

Activity 6

Let $D(I,\mathbb{R})$ be the set of functions differentiable on I.

1. If $g\in D(I,\mathbb{R})$ and $g(x)\neq 0$, using the definition of derivative find derivative of $\frac{1}{g}$
2. If $f\in D(I,\mathbb{R})$, $g\in D(I,\mathbb{R})$,and $g(x)\neq 0$, using the result in 1 and result for derivative of a product find $\frac{f}{g}$

Derivative of the reciprocal function

From activity 4,

Let $D(I,\mathbb{R})$ be the set of functions differentiable on I.

If $f\in D(I,\mathbb{R})$, then $\frac{1}{f}\in D(I,\mathbb{R})$ $f(x)\neq 0$. Moreover

$$\frac{d}{dx}\left(\frac{1}{f}\right)=-\frac{\frac{df}{dx}}{f^2}.$$

From activity 4,

If f and g are differentiable at x and if $g(x) \neq 0$, then the quotient $\frac{f}{g}$ is differentiable at x and

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

Example 15

Find the derivative of $\frac{1}{f(x)}$ if $f(x) = 6x$

Solution

$$\left(\frac{1}{f(x)}\right)' = \left(\frac{1}{6x}\right)' = \frac{-(6x)'}{(6x)^2} = \frac{-6}{36x^2} = \frac{-1}{6x^2}$$

Example 16

Find the derivative of $y = \frac{3}{5-2x}$

Solution

$$\frac{dy}{dx} = -\frac{\frac{d}{dx}(5-2x)}{(5-2x)^2} = \frac{2}{(5-2x)^2}$$

Example 17

Find the derivative of $f(x) = \frac{2x^2+3x}{4x^3+x+1}$

$$f'(x) = \left(\frac{2x^2+3x}{4x^3+x+1}\right)'$$

$$= \frac{(2x^2+3x)'(4x^3+x+1) - (2x^2+3x)(4x^3+x+1)'}{(4x^3+x+1)^2}$$

$$= \frac{(2x^2+3x)'(4x^3+x+1) - (2x^2+3x)(4x^3+x+1)'}{(4x^3+x+1)^2}$$

$$= \frac{(4x+3)(4x^3+x+1) - (2x^2+3x)(12x^2+1)}{(4x^3+x+1)^2}$$

$$= \frac{16x^4+4x^2+4x+12x^3+3x+3-24x^4-2x^2-36x^3-3x}{(4x^3+x+1)^2}$$

$$= \frac{-8x^4-24x^3+2x^2+4x+3}{(4x^3+x+1)^2}$$

Example 18

Given that $y=\dfrac{x^2-4}{x^2+4}$, find $\dfrac{dy}{dx}$.

We apply quotient rule with $f(x)=x^2-4$ and $g(x)=x^2+4$:

$$\frac{dy}{dx}=\frac{(x^2+4)\cdot 2x-(x^2-4)\cdot 2x}{(x^2+4)^2}$$

$$=\frac{2x^3+8x-2x^3+8x}{(x^2+4)^2}$$

$$=\frac{16x}{(x^2+4)^2}$$

Example 19

Let $f(x)=\dfrac{x^2+2x+5}{\sqrt{4x+1}}$, find its derivative.

Solution

$$f'(x)=\left(\frac{x^2+2x+5}{\sqrt{4x+1}}\right)'$$

$$=\frac{(x^2+2x+5)'\sqrt{4x+1}-(x^2+2x+5)\left(\sqrt{4x+1}\right)'}{\left(\sqrt{4x+1}\right)^2}$$

$$=\frac{(2x+2)\sqrt{4x+1}-(x^2+2x+5)\dfrac{4}{2\sqrt{4x+1}}}{4x+1}$$

$$=\frac{(2x+2)(4x+1)-2x^2-4x-10}{(4x+1)\sqrt{4x+1}}$$

$$=\frac{8x^2+2x+8x+2-2x^2-4x-10}{(4x+1)\sqrt{4x+1}}$$

$$=\frac{6x^2+6x-8}{(4x+1)\sqrt{4x+1}}$$

Example 20

Find derivative of $f(x)=\dfrac{1}{\sqrt{4x^2+3}}-(x^3+5x+2)^5$

$$f(x)=\left[\frac{1}{\sqrt{4x^2+3}}\right]'-\left[\left(x^3+5x+2\right)^5\right]'$$

$$=\frac{\frac{8x}{2\sqrt{4x^2+3}}}{4x^2+3}-5\left(x^3+5x+2\right)^4(3x^2+5)$$

$$=-\frac{4x}{\left(4x^2+3\right)\sqrt{4x^2+3}}-(x^3+5x+2)^4(15x^2+25)$$

Exercise 6

Find the derivative of the following functions

1. $f(x)=\frac{3x^6+3x^4+6x-6}{2x^2+4x+1}$
2. $f(x)=\frac{1}{x^3+2x^2+6}$

e) Composite function

Activity 7

Given the function $f(x)=x^2+3x-4$ and $g(x)=x+1$. Find

1. $f[g(x)]$
2. $\left(f\left[g(x)\right]\right)'$
3. $f'(x)$
4. $f'[g(x)]$
5. $g'(x)$
6. $f'[g(x)].g'(x)$

Compare results in 2 and 6

Derivative of a composite function: Chain rule

From activity 5:

If $f\in D(I,\mathbb{R})$ and $g\in D(I,\mathbb{R})$, then $g\circ f\in D(I,\mathbb{R})$. In addition $(g\circ f)'=g'(f)f'$

Moreover, if $y=f\left[g(x)\right]$ and $u=g(x)$ then $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

Another notation of Chain rule states

$$\frac{d}{dx}\left[f\left[g(x)\right]\right]=f'\left[g(x)\right]g'(x)$$

Example 21

Find the derivative of *fog* if $f(x)=x^2+3x+3$ and

$$g(x)=\frac{x^2+1}{x}$$

Solution

$$(fog)'(x)=f'[g(x)]g'(x)$$

$$=\left[2\left(\frac{2x+1}{x}\right)+3\right]\left(\frac{2x+1}{x}\right)$$

$$=\frac{4x+2+3x}{x}\left(\frac{2x-2x-1}{x2}\right)$$

$$=\frac{7x+2}{x}\left(\frac{-1}{x^2}\right)$$

$$=\frac{-7x-2}{x^3}$$

Example 22

Find the derivative of ***fog*** if $f(x)=3x-4$ and $g(x)=x-3$

Solution

$$g'(x)=1,\ f'(x)=3$$

$$f'[g(x)]=3$$

$$(fog)'(x)=f'[g(x)]g'(x)=3(1)=3$$

Exercise 7

In each of the following find $(f\circ g)'(x)$

1. $f(x)=2x-4$ and $g(x)=x+3$
2. $f(x)=\dfrac{x-5}{2}$ and $g(x)=x^2+3$
3. $f(x)=x^2+4x+3$ and $g(x)=x^2+1$
4. $f(x)=x^5-30$ and $g(x)=7$
5. $f(x)=4$ and $g(x)=\dfrac{x^2+3x+3}{x^2-4}$

Successive derivatives

Activity 8

Consider the function $f(x)=x^6+x^5+3x^3-2x^2+x+8$. Find

1. $f'(x)$
2. The derivative of the function obtained in 1.
3. The derivative of the function obtained in 2.
4. The derivative of the function obtained in 3.
5. The derivative of the function obtained in 4.

We have seen that the derivative of $y=f(x)$ is in general also a function of x. This new function can be also differentiable, in which case the derivative of the first derivative is called the **second derivative** of the original function. It is written in several ways:

$$f''(x)=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{dy'}{dx}=y''=D^2\left(f(x)\right)=D^2f(x).$$

The symbol D^2 means the operation of differentiation is performed twice.

Similarly, the derivative of the second derivative is called the **third derivative** and so on.

Thus, if for example $y=3x^4$ then,

$\frac{dy}{dx}=12x^3$, $\frac{d}{dx}\left(\frac{dy}{dx}\right)=36x^2$, $\frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right]=72x$ And so on.

The successive derivatives of a function f are **higher order derivatives** of the same function.

We denote higher order derivatives of the same function as follows:

The second derivative is:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}=f''(x)=y''$$

The third derivative is:

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d^3y}{dx^3}=f'''(x)=y'''$$

And the n^{th} derivative is:

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)=\frac{d^ny}{dx^n}=f^{(n)}(x)=y^{(n)}$$

Example 22

Successive derivatives of $y=x^n \quad n\in\mathbb{R}$

$$y'=nx^{n-1}$$
$$y''=n(n-1)x^{n-2}$$
$$y'''=n(n-1)(n-2)x^{n-3}$$
$$y^{(n)}=n(n-1)(n-2)...x^{n-n}=n(n-1)(n-2)....1=n!$$

Thus, if $y=x^n \quad n\in\mathbb{R},\ y^{(n)}=n!$

Example 23

Given $y=x^4-3x+4$. Let us find $\frac{d^5y}{dx^5}$

Solution

$$y'=4x^3-3$$
$$y''=12x^2$$
$$y'''=24x$$
$$y^{(4)}=24$$
$$y^{(5)}=0$$

Thus, $y^{(5)}=0$

Exercise 8

1. Find $\frac{d^4y}{dx^4}$ if $y=4x^7+6x+8$
2. Find $\frac{d^5y}{dx^5}$ if $y=12x+6$
3. Find $\frac{d^2y}{dx^2}$ if $y=\frac{x+1}{x-2}$
4. Find $\frac{d^4y}{dx^4}$ if $y=\frac{x^2-4}{x+2}$

3. Applications of differentiation

Equation of tangent line and normal line

Activity 9

Consider the function $f(x)=-x^3+3x$ and the line $y=3x$ passing through point $(0,0)$.

1. Show that is the intersection of $f(x)=-x^3+3x$ and $y=3x$
2. Find $f'(0)$
3. Compare the result in 2. And the gradient of the given line.

Tangent line

The slope of the tangent line of $y=f(x)$ at $(x_0, f(x_0)=y_0)$ is given by $f'(x_0)=\frac{y-y_0}{x-x_0}$.

Then, the equation of the tangent line is

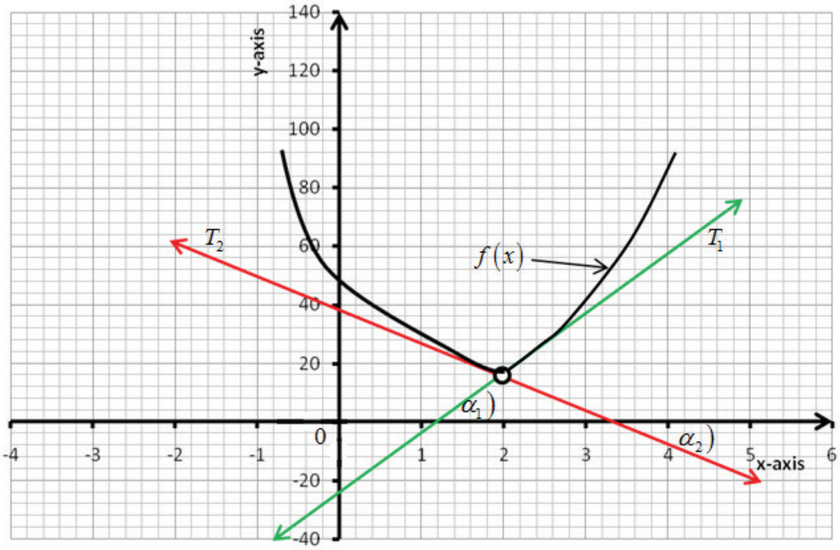
$$T \equiv y-y_0=f'(x_0)(x-x_0)$$

Remark

Remember that the function $f(x)$ can have distinct right-hand and left- hand derivatives at point x_0; that is, $f'(x_0^-)\neq f'(x_0^+)$.

In this case we say that the point x_0 is a **sharp.** The curve has no tangent line at x_0. Centrally, it has a half tangent at the left and another at the right with different slopes.

See the following figure.

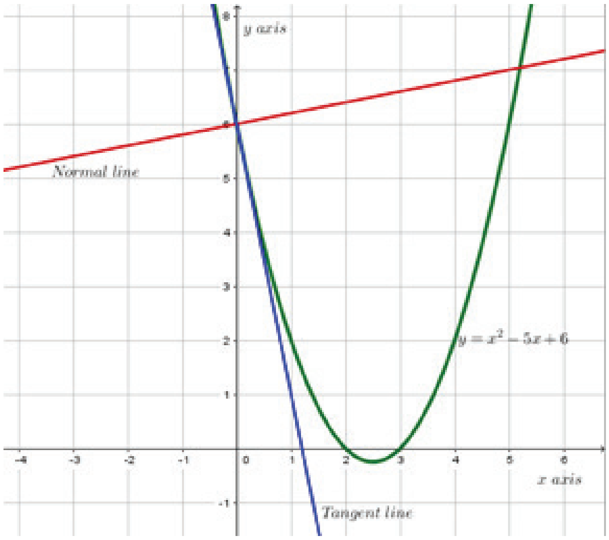


$$\tan\alpha_1 = \lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0} \text{ And } \tan\alpha_2 = \lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}$$

Normal line

We call **normal line** to the curve at point (x_0, y_0) the perpendicular line to the tangent line of the curve at point (x_0, y_0). Its equation is of the form

$$N \equiv y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$$



Example 24

Given the parabola $f(x)=x^2$

a) Find the point where the tangent line is parallel to the bisector of the first quadrant.

b) Find the tangent line to the curve of this function at point $(2,4)$

Solution

a) The bisector of the first quadrant has the equation $y=x$, so its slope is $m=1$.

Since the two lines are parallel, they have the same slope.

So $f'(x_0)=1$.

Since the slope of the tangent line to the curve is equal to the derivative at $x=x_0$,

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0}\frac{(x_0+h)^2-x_0^2}{h} \\ &= \lim_{h\to 0}\frac{x_0^2+2x_0h+h^2-x_0^2}{h} \\ &= \lim_{h\to 0}\frac{2x_0h+h^2}{h} \\ &= \lim_{h\to 0}(2x_0+h) \\ &= 2x_0 \end{aligned}$$

But $f'(x_0)=1 \Rightarrow 2x_0=1 \Rightarrow x_0=\frac{1}{2}$ and $y_0=f\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^2=\frac{1}{4}$.

Thus, the needed point is $\left(\frac{1}{2},\frac{1}{4}\right)$.

b) The given point is $(2,4)$, then $x_0=2,\ y_0=4$. $f'(x_0)=2x_0$

$\Rightarrow f'(2)=4$

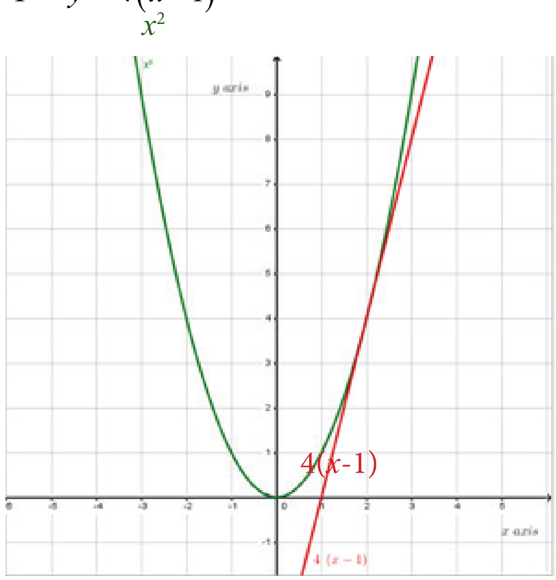
The tangent line is

$$T \equiv y-4=4(x-2)$$

$$T \equiv y-4=4x-8$$

$$T \equiv y=4x-4$$

$$T \equiv y=4(x-1)$$



Normal line

We call **normal line** to the curve at point (x_0, y_0), the perpendicular line to the tangent line of the curve at point (x_0, y_0). Its equation is of the form

$$N \equiv y-y_0=-\frac{1}{f'(x_0)}(x-x_0)$$

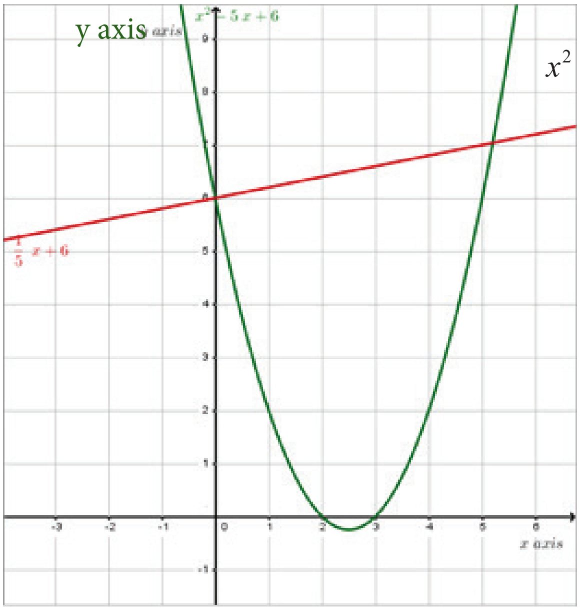
Example 25

Let us determine the equation of the normal line to the curve with equation $y=x^2-5x+6$ at point with abscissa $x=0$.

$f'(x)=2x-5$, $f'(0)=-5$, $f(0)=6$

The equation of normal line is $N \equiv y-6=-\frac{1}{-5}(x-0)$ or

$$N \equiv y=\frac{1}{5}x+6$$



Rates of change

The purpose here is to remind ourselves one of the more important applications of derivatives. That is the fact that $f'(x)$ represents the rate of change of $f(x)$.

If (x_0, y_0) are points on the graph of $y = f(x)$, then we define $m = \frac{y_1 - y_0}{x_1 - x_0}$ to be the average rate at which y changes with x over the interval $[x_0, x_1]$.

If $y = f(x)$ and $f(x)$ is differentiable at x_0, then we define $n = \frac{dy}{dx}|_{x=x_0}$ to be the instantaneous rate at which y changes with $x = x_0$

Example 26

For the curve $y = x^2 + 1$. Let us find the average rate of change of y with x over the interval $[3,5]$ and the instantaneous rate of change of y with x at point $x = 3$.

Here $x_0 = 3$ and $x_1 = 5$

$y_0 = (3)^2 + 1 = 10, y_1 = (5)^2 + 1 = 26$

So average rate of y over $[3,5]$ is $\frac{26-10}{5-3} = \frac{16}{2} = 8$. Thus, on the average y increases 8 units for each unit increase in x over the interval $[3,5]$.

$\frac{dy}{dx} = f'(x) = 2x$, So instantaneous rate of change of y at $x = 3$ Is $\frac{dy}{dx}|_{x=3} = 2x|_{x=3} = 6$.

Thus, at point $x = 3$, y is increasing 6 times as fast as x.

Example 27

Let us find all points where the function $f(x) = \sin x$ is not changing.

This function will not be changing if the rate of change is zero. Then we need to determine where the derivative is zero.

$\frac{dy}{dx} = f'(x) = \cos x$ And $\cos x = 0$ for $x = \pm\frac{\pi}{2} + 2k\pi$ or simply $x = \frac{\pi}{2} + k\pi$. Thus, $f(x) = \sin x$ is not changing if $x = \frac{\pi}{2} + k\pi$ $k \in \mathbb{Z}$.

Critical points

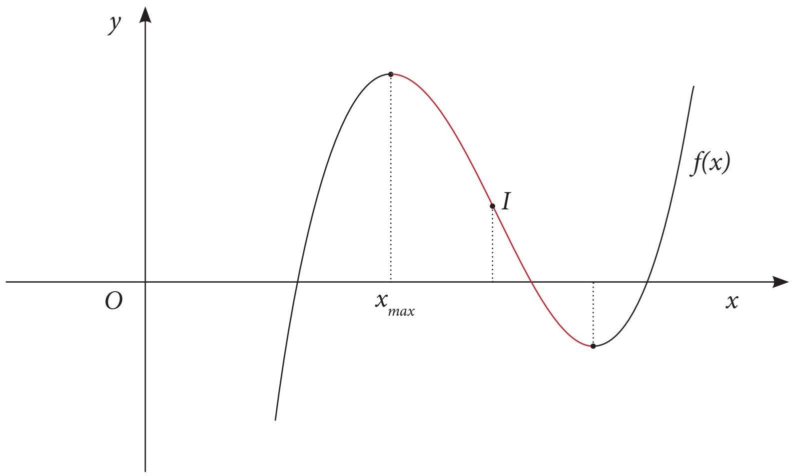
We say that $x = c$ is a critical point for the function $f(x)$ if $f(c)$ exists and if either one of the following is true

- $f'(c) = 0$ Or $f'(c)$ does not exist.

Note that we require that $f(c)$ exists in order for $x = c$ to actually be a critical point.

This is an important and often overlooked point.

The critical point $x = c$ where $f'(c) = 0$ is called **stationary point** of f.



Example 28

Let us determine all the critical points for the function

$f(x) = 6x^5 + 33x^4 - 30x^3 + 100$

We first need the derivative of the function in order to find the critical points and so let us get that and notice that we will factor it as much as possible to make our life easier when we go to find the critical points.

$$\begin{aligned} f'(x) &= \left(6x^5 + 33x^4 - 30x^3 + 100\right)' \\ &= 30x^4 + 132x^3 - 90x^2 \\ &= 6x^2\left(5x^2 + 22x - 15\right) \\ &= 6x^2(5x-3)(x+5) \end{aligned}$$

Now, our derivative is a polynomial and so will exist everywhere.

Therefore the only critical points will be those values of x which make the derivative zero. So, we must solve $6x^2(5x-3)(x+5) = 0$

Because this is the factored form of the derivative, it's pretty easy to identify the three critical points. They are, $x = -5, x = 0, x = \frac{3}{5}$

Example 29

Let us determine all the critical points for the function

$$f(x)=\sqrt[3]{x^2}\,(2x-1)$$

To find the derivative, it's probably easiest to do a little simplification before we actually differentiate. Let's multiply the root through the parenthesis and simplify as much as possible.

This will allow us to avoid using the product rule when taking the derivative.

$$f(x)=x^{\frac{2}{3}}(2x-1)=2x^{\frac{5}{3}}-x^{\frac{2}{3}}$$

Now differentiate

$$f'(x)=\frac{10}{3}x^{\frac{2}{3}}-\frac{2}{3}x^{\frac{-1}{3}}=\frac{10x^{\frac{2}{3}}}{3}-\frac{2}{3x^{\frac{1}{3}}}=\frac{10x-2}{3x^{\frac{1}{3}}}$$

We will need to be careful with this problem. When faced with a negative exponent it is often best to eliminate the minus sign in the exponent as we did above. This is not really required but it can make our life easier on occasion if we do that.

This derivative is zero if numerator is zero. That is, $10x-2=0\Rightarrow x=\frac{1}{5}\cdot$

But this derivative is not defined at $x=0$ and so there are two critical points $x=0$ and $x=\frac{1}{5}$

Example 30

Let us determine all the critical points of the function

$$g(t)=\frac{t^2+1}{t^2-t-6}$$

$$g'(t)=\left(\frac{t^2+1}{t^2-t-6}\right)'$$

$$=\frac{2t\left(t^2-t-6\right)-\left(t^2+1\right)\left(2t-1\right)}{\left(t^2-t-6\right)^2}$$

$$=\frac{2t^3-2t^2-12t-2t^3+t^2-2t+1}{\left(t^2-t-6\right)^2}$$

$$=\frac{-t^2-14t+1}{\left(t^2-t-6\right)^2}$$

Now, we have two issues to deal with. First the derivative will not exist if there is division by zero in the denominator. So we need to solve, $t^2-t-6=(t-3)(t+2)=0$

So, we can see from this that the derivative will not exist at $t=3$ and $t=-2$. However, these are not critical points since the function will also not exist at these points. Recall that in order for a point to be a critical point the function must actually exist at that point.

At this point, we have to be careful. The numerator does not factor, but that does not mean that there are not any critical points where the derivative is zero. We can use the quadratic formula on the numerator to determine if the fraction as a whole is ever zero.

Now, $-t^2-14t+1=-\left(t^2+14t-1\right)=0 \Leftrightarrow t^2+14t-1=0$

$\Delta=196+4=200$

$$t=\frac{-14\pm10\sqrt{2}}{2}=-7\pm5\sqrt{2}$$

Thus, the critical point are $t=-7+5\sqrt{2}$ and $t=-7-5\sqrt{2}$.

Finding absolute extrema

Absolute extrema are the largest and smallest the function will ever be. To find absolute extrema of function $f(x)$ on $[a,b]$ follow the following steps:

a) Verify that the function is continuous on the interval $\left[a,b\right]$.

b) Find all critical points of $f\left(x\right)$ that are in the interval $\left[a,b\right]$. This makes sense if you think about it. Since we are only interested in what the function is doing in this interval, we do not care about critical points that fall outside the interval.

c) Evaluate the function at the critical points found in a) above and the end points.

d) Identify the absolute extrema.

Example 31

Let us determine the absolute extrema for the following function

$f\left(x\right)=2x^3+3x^2-12x+4$ On $\left[-4,2\right]$.

First notice that this is a polynomial and so is continuous everywhere and in particular is then continuous on the given interval.

Now, we need to get the derivative so that we can find the critical points of the function.

$$\begin{aligned} f'\left(x\right) &= 6x^2+6x-12 \\ &= 6\left(x^2+x-2\right) \\ &= 6\left(x+2\right)\left(x-1\right) \end{aligned}$$

It looks like we will have two critical points, $x=-2$ and $x=1$. Note that we actually want something more than just the critical points. We only want the critical points of the function that lie in the interval in question. Both of these do fall in the interval as so we will use both of them.

Now we evaluate the function at the critical points and the end points of the interval.

$f(-2)=24$ $\quad$ $f(1)=-3$

$f(-4)=-28$ $\quad$ $f(2)=8$

From this list we see that the absolute maximum of $f(x)$ is 24 and it occurs at $x=-2$ (a critical point) and the absolute minimum of $f(x)$ is -28 which occurs $x=-4$ (an endpoint).

Finding relative extrema

First derivative test:

Suppose f is continuous at a critical point x_0;

a) If $f'(x_0)>0$ on an open interval extending left from x_0 and $f'(x_0)<0$ on an open interval extending right from x_0, then f has a relative maximum at x_0.

b) If $f'(x_0)<0$ on an open interval extending left from x_0 and $f'(x_0)>0$ on an open interval extending right from x_0, then f has a relative minimum at x_0.

c) If $f'(x)$ has the same sign (either $f'(x_0)>0$ or $f'(x_0)<0$) on an open interval extending left and on an open interval extending right from x_0, then f does not have a relative extremum at x_0.

i.e, the relative extrema of a continuous function occur at those critical points where the first derivative changes sign.

Example 32

Locate the relative extrema of $f(x)=3x^{\frac{5}{3}}-15x^{\frac{2}{3}}$

$f'(x)=5x^{\frac{2}{3}}-10x^{\frac{-1}{3}}=5x^{\frac{-1}{3}}(x-2)$

Since $f'(x)$ does not exist at $x=0$ but $f(0)$ exists and $f'(x)=0$ if $x=2$, the critical point are 0 and 2. Sign table of $f'(x)$

x		0		2	
$x-2$	-	-	-	0	+
$x^{\frac{-1}{3}}$	-	0	+	+	+
$f'(x)$	+	$\|\|$ ∞	-	0	+

There is a relative maximum at 0 and a relative minimum at 2.

Example 33

Locate the relative extrema of $f(x)=x^3-3x^2+3x-1$

$f'(x)=3x^2-6x+3=3(x-1)^2$

Solving for $f'(x)=0$, yields $x=1$ as the only critical point.

Since $f'(x)=3(x-1)^2\geq 0$ for all x, $f'(x)$ does not change the sign. Thus, no relative extremum exists.

Second derivative test:

Suppose that $f(x)$ is twice differentiable at stationary point x_0

a) If $f''(x_0)>0$, then there is a relative minimum at x_0

b) If $f''(x_0)<0$, then there is a relative maximum at x_0

Note that the second derivative test does not apply if

$f''(x_0)=0$

Example 34

Locate and describe the relative extrema of $f(x)=x^4-2x^2$

$f'(x)=4x^3-4x=4x(x^2-1)=4x(x-1)(x+1)$

Solving for $f'(x)=0$, yields the stationary points -1, 0 and 1.

$f''(-1)=8>0$

$f''(0)=-4<0$

$f''(1)=8>0$

There is a relative maximum at $x=0$ and there are relative minimum at $x=-1$ and $x=1$.

Extreme value theorem

Suppose that $f(x)$ is continuous on the interval $[a,b]$ then there are two numbers $a \le c, d \le b$ so that $f(c)$ is an absolute maximum for the function and $f(d)$ is an absolute minimum for the function.

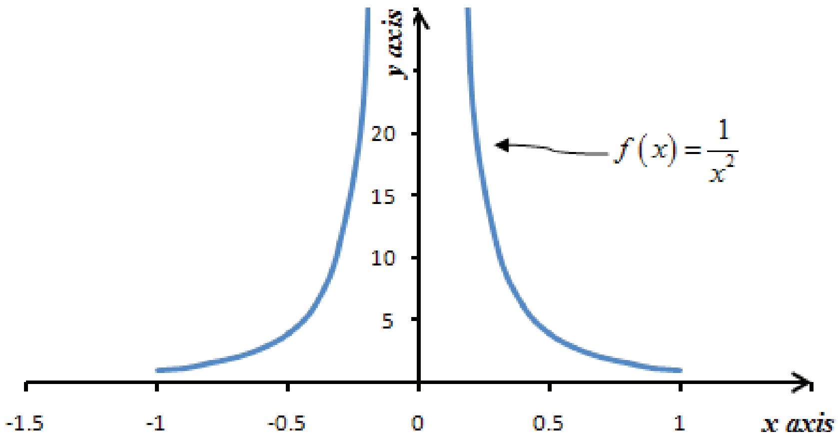
So, if we have a continuous function on an interval $[a,b]$ then we are guaranteed to have both an absolute maximum and an absolute minimum for the function somewhere in the interval. The theorem doesn't tell us where they will occur or if they will occur more than once, but atleast it tells us that they do exist somewhere. Sometimes, all that we need to know is that they do exist. This theorem doesn't say anything about absolute extrema if we aren't working on an interval.

The requirement that a function be continuous is also required in order for us to use the theorem.

Example 35

Consider the case of $f(x)=\frac{1}{x^2}$ on $[-1,1]$.

Here's the graph.



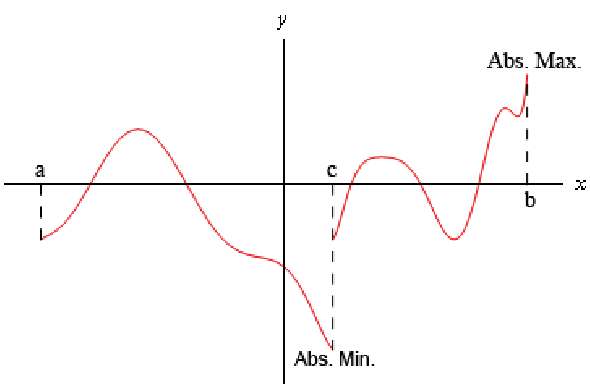
This function is not continuous at $x=0$ as we move in towards zero the function is approaching infinity. So, the function does not have an absolute maximum. Note that it has an absolute minimum however. In fact the absolute minimum occurs twice at both $x=-1$ and $x=1$.

If we changed the interval a little to say, $f(x)=\frac{1}{x^2}$ on $\left[\frac{1}{2},1\right]$ the function would now have both absolute extrema. We may only run into problems if the interval contains the point of discontinuity. If it doesn't, then the theorem will hold.

We should also point out that just because a function is not continuous at a point, that doesn't mean that it won't have both absolute extrema in an interval that contains that point.

Example 36

Consider the following graph



This graph is not continuous at $x=c$, yet it does have both an absolute maximum (at $x=b$) and an absolute minimum (at $x=c$). Also note that, in this case one of the absolute extrema occurred at the point of discontinuity, but it doesn't need to. The absolute minimum could just have easily been at the other end point or at some other point interior to the region. The point here is that this graph is not continuous and yet does have both absolute extrema

The point of all this is that we need to be careful to only use the extreme value theorem when the conditions of the theorem are met and not misinterpret the results if the conditions are not met.

Note

In order to use the extreme value theorem, we must have an interval and the function must be continuous on that interval. If we don't have an interval and/or the function isn't continuous on the interval, then the function may or may not have absolute extrema.

Fermat's theorem

If $f(x)$ has a relative extrema at $x=c$ and $f'(c)$ exists then $x=c$ is a critical point of $f(x)$. In fact, it will be a critical point such that $f'(c)=0$.

Note that we can say that $f'(c)=0$ because we are also assuming that $f'(c)$ exists.

This theorem tells us that there is a nice relationship between relative extrema and critical points. In fact it will allow us to get a list of all possible relative extrema. Since a relative extrema must be a critical point, the list of all critical points will give us a list of all possible relative extrema.

Example 37

Consider the case of $f(x)=x^2$. We saw that this function had a relative minimum at $x=0$ in several earlier examples. So according to fermat's theorem $x=0$ should be a critical point. The derivative of the function is, $f'(x)=2x$. Sure enough $x=0$ is a critical point.

Be careful not to misuse this theorem. It doesn't say that a critical point will be a relative extrema. To see this, consider the following example.

Example 38

Consider the function $f(x)=x^3$ then $f'(x)=3x^2$

Clearly $x=0$ is a critical point. However this function has no relative extrema of any kind. So, critical points do not have to be relative extrema.

Also note that this theorem says nothing about absolute extrema. An absolute extrema may or may not be a critical point.

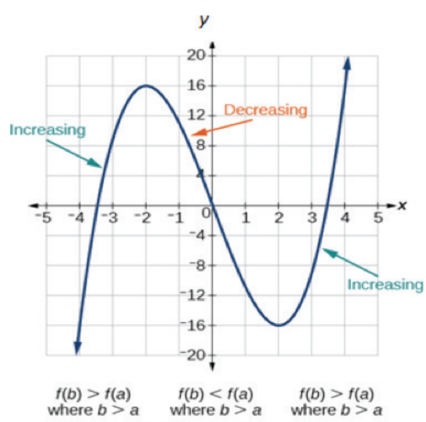
Increasing and decreasing of a function

In the previous lines we saw how to use the derivative to determine the absolute minimum and maximum values of a function. However, there is a lot more information about a graph that can be determined from the first derivative of a function.The main idea we will be looking at here, we will be identifying all the relative extrema of a function.

We know from our work in previous lines that the first derivative, $f'(x)$, is the rate of change of the function. We used this idea to identify where a function was increasing, decreasing or not changing. Let us see definitions:

Given any x_1 and x_2 from an interval *i* with $x_1 < x_2$ if $f(x_1) < f(x_2)$ then $f(x)$ is **increasing** on *I*.

Given any x_1 and x_2 from an interval *I* with $x_1 < x_2$ if $f(x_1) > f(x_2)$ then $f(x)$ is **decreasing** on *I*.



Now, recall that earlier we constantly used the idea that if the derivative of a function was positive at a point then the function was increasing at that point and if the derivative was negative at a point then the function was decreasing at that point. We also used the fact that if the derivative of a function was zero at a point then the function was not changing at that point. We used these ideas to identify the intervals in which a function is increasing and decreasing. This can be summarised in the following fact.

Fact

- If $f'(x) > 0$ for every x on some interval i, then $f(x)$ is increasing on the interval.
- If $f'(x) < 0$ for every x on some interval i, then $f(x)$ is decreasing on the interval.
- If $f'(x) = 0$ for every x on some interval i, then $f(x)$ is constant on the interval.

Example 39

Let us determine all intervals where the following function is increasing or decreasing. $f(x) = -x^5 + \frac{5}{2}x^4 + \frac{40}{3}x^3 + 5$

To determine if the function is increasing or decreasing we will need the derivative.

$$\begin{aligned} f'(x) &= -5x^4 + 10x^3 + 40x^2 \\ &= -5x^2\left(x^2 - 2x - 8\right) \\ &= -5x^2(x-4)(x+2) \end{aligned}$$

From the factored form of the derivative, we see that we have three critical points: $x = -2$, $x = 0$, and $x = 4$. We will need these in a bit.

We now need to determine where the derivative is positive and where it's negative. Since the derivative is a polynomial, it is continuous and so we know that the only way for it to change signs is to first go through zero.

In other words, the only place that the derivative may change signs is at the critical points of the function. We have now got another use for critical points. So, we will build sign table of $f'(x)$, graph the critical points and pick test points from each region to see if the derivative is positive or negative in each region.

x	$-\infty$		-2		0		4		$+\infty$
$f'(x)$		-	0	+	0	+	0	-	
$f(x)$	$+\infty$	↘	$f(-2)$	↗	$f(0)$	↗	$f(4)$	↘	$-\infty$

We have

Increase (symbolized by the arrow ↗):

$-2 < x < 0$ and $0 < x < 4$

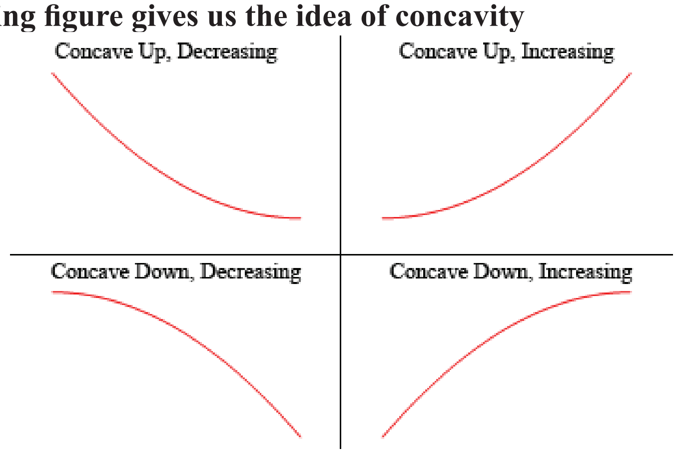
Decrease (symbolized by the arrow ↘):

$x < 2$ and $x >$

Concavity of a function

In the lines, we saw how we could use the first derivative of a function to get some information about the graph of a function. In following lines, we are going to look at the information that the second derivative of a function can give us about the graph of a function.

The following figure gives us the idea of concavity

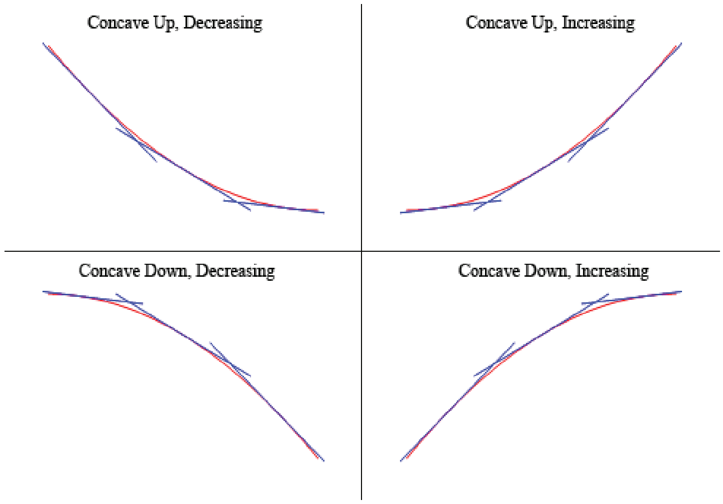


So a function is **concave up** if it "opens" up and the function is **concave down** if it "opens" down. Notice as well that concavity has nothing to do with increasing or decreasing. A function can be concave up and either increasing or decreasing. Similarly, a function can be concave down and either increasing or decreasing.

Given the function $f(x)$ then

- $f(x)$ Is concave up on an interval i if all of the tangents to the curve on i are below the graph of $f(x)$.
- $f(x)$ Is concave down on an interval i if all of the tangents to the curve on i are above the graph of $f(x)$.

To show that the graphs above do in fact have concavity claimed above here is the figure again (blown up a little to make things clearer).



So, as you can see, in the two upper graphs all of the tangent lines sketched in are all below the graph of the function and these are concave up. In the lower two graphs all the tangent lines are above the graph of the function and these are concave down.

There's one more definition that we need to get out of the way.

A point $x = c$ is called an **inflection point** if the function is continuous at the point and the concavity of the graph changes at that point.

Now that we have all the concavity definitions out of the way, we need to bring the second derivative into the mix. The following fact relates the second derivative of a function to its concavity.

Fact

Given the function $f(x)$ then,

- If $f''(x) > 0$ for all x in some interval I then $f(x)$ is concave up on I.

- If $f''(x) < 0$ for all x in some interval I then $f(x)$ is concave down on I.

Notice that this fact tells us that a list of possible inflection points will be those points where the second derivative is zero or doesn't exist. Be careful, however, to not make the assumption that just because the second derivative is zero or doesn't exist that the point will be an inflection point.

We will only know that it is an inflection point once we determine the concavity on both sides of it. It will only be an inflection point if the concavity is different on both sides of the point.

Example 40

Let us find where the function $f(x) = x^3 - 3x^2$ is concave up or down.

We need the second derivative

$f'(x) = 3x^2 - 6x$,

$f''(x) = 6x - 6$

Sign of $f''(x)$

x		1	
$f''(x)$	-	0	+

Thus, $f(x)$ is concave up if $x>1$ and

$f(x)$ Is concave down if $x<1$

Rolle's theorem

Suppose that $f(x)$is a function that satisfies all of the following.

- $f(x)$ Is continuous on the closed interval $[a,b]$.
- $f(x)$ Is differentiable on the open interval (a,b).
- $f(a)=f(b)$.

Then, there is a number c such that $a<c<b$ and $f'(c)=0$.
Or, in other words $f(x)$has a critical point in (a,b).

Example 41

Consider the function $f(x)=x^2-1$ on $[-1,1]$

This function is continuous on $[-1,1]$ and differentiable on $(-1,1)$

Moreover $f(-1)=f(1)=0$.
Then from rolle's theorem, we must get a number c such that $-1<c<1$ and $f'(c)=0$.

The first derivative is $f'(x)=2x$ and $f'(x)=0$ for $x=0$ and we see that $-1<0<1$.

Mean value theorem

Suppose that $f(x)$is a function that satisfies both of the following;

- $f(x)$ Is continuous on the closed interval $[a,b]$.
- $f(x)$ Is differentiable on the open interval (a,b).

Then, there is a number c such that $a<c<b$ and

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Or, $f(b)-f(a)=f'(c)(b-a)$.
Note that the mean value theorem doesn't tell us what c is. It only tells us that there is atleast one number c that will satisfy the conclusion of the theorem.

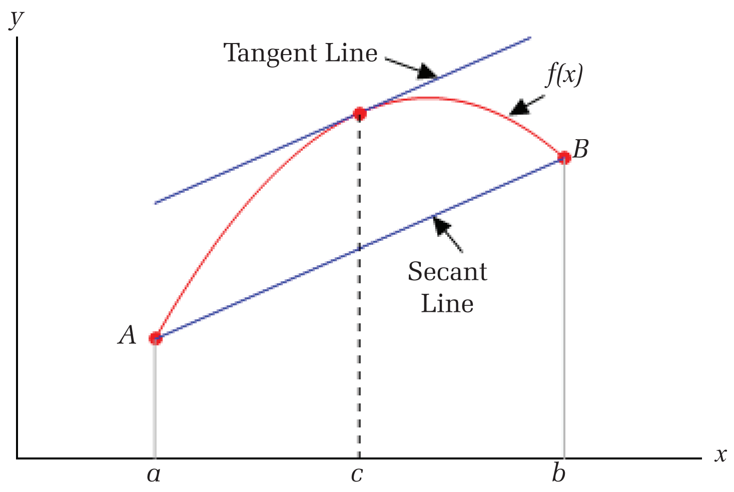
Also note that if $f(a)=f(b)$, we can think of rolle's theorem as a special case of the mean value theorem.

Geometrical interpretation of the mean value theorem

First define $A=(a,f(a))$ and $B=(b,f(b))$ and then we know from the mean value theorem that there is a c such that $a<c<b$ and that $f'(c)=\frac{f(b)-f(a)}{b-a}$.
Now, if we draw in the secant line connecting a and b then we can know that the slope of the secant line is,
$\frac{f(b)-f(a)}{b-a}$.
Likewise, if we draw in the tangent line to $f(x)$ at $x=c$ we know that its slope is $f'(c)$.

What the mean value theorem tells us is that these two slopes must be equal or in other words the secant line connecting A and B and the tangent line at $x=c$ must be parallel.

We can see this in the following sketch.



Example 42

Let us determine all the numbers c which satisfy the conclusions of the mean value theorem for the function $f(x)=x^3+2x^2-x$ on $[-1,2]$.

There isn't really a whole lot to this problem other than to notice that since $f'(x)$ is a polynomial, it is both continuous and differentiable (i.E, the derivative exists) on the interval given.

First derivative, $f'(x)=3x^2+4x-1$

Now, to find the numbers that satisfy the conclusions of the mean value theorem all we need to do is plug this into the formula given by the mean value theorem.

$$f'(x)=\frac{f(2)-f(-1)}{2-(-1)}$$

$$\Leftrightarrow 3c^2+4c-1=\frac{14-2}{3}=4 \Leftrightarrow 3c^2+4c-1=4 \Leftrightarrow 3c^2+4c-5=0$$

$$\Delta=16+60=76$$

$$c=\frac{-4\pm2\sqrt{19}}{6}=\frac{-2\pm\sqrt{19}}{3}$$

Thus, the values of c which satisfy the conclusions of the mean value theorem for the function $f(x)=x^3+2x^2-x$ on $[-1,2]$ is $\frac{-2+\sqrt{19}}{3}$. The value $\frac{-2-\sqrt{19}}{3}$ is excluded since it is not an element of the given interval.

Let us see a couple of nice facts.

Fact 1

If $f'(x)=0$ for all x in an interval (a,b) then $f(x)$ is constant on (a,b).

Fact 2

If $f'(x) = g'(x)$ for all x in an interval (a,b) then in this interval we have $f(x) = g(x) + c$ where c is a constant.

Note that in both of these facts we are assuming the functions are continuous and differentiable on the interval $[a,b]$.

L'hôpital's rule

Back on the section of limits, we saw methods for dealing with the following limits:

$$\lim_{x\to 4}\frac{x^2-16}{x-4} \qquad \lim_{x\to\infty}\frac{4x^2-5x}{1-3x^2}$$

In the first limit if we plugged in $x = 4$, we would get $\frac{0}{0}$ and in the second limit if we plugged in infinity, we would get $\frac{\infty}{-\infty}$ which are the indeterminate forms. In both of these cases, there are competing interests or rules and it is not clear which one will win out.

For the two limits above we work them out as follows.

$$\lim_{x\to 4}\frac{x^2-16}{x-4} = \lim_{x\to 4}\frac{(x+4)(x-4)}{x-4}$$
$$= \lim_{x\to 4}(x+4)$$
$$= 8$$

$$\lim_{x\to\infty}\frac{4x^2-5x}{1-3x^2} = \lim_{x\to\infty}\frac{x^2\left(4-\frac{5}{x}\right)}{x^2\left(\frac{1}{x^2}-3\right)}$$
$$= \lim_{x\to\infty}\frac{4-\frac{5}{x}}{\frac{1}{x^2}-3}$$
$$= -\frac{4}{3}$$

In the first case we simply factored, canceled and took the limit and in the second case we factored out an x^2 from both the numerator and the denominator and took the limit. Notice as well that none of the competing interests or rules in these cases won out!

There is another method that can help us to evaluate such limits and it is called l'hôpital's rule. It tells us that if we have an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ all we need to do is to differentiate the numerator and the denominator and then take the limit.

That is, suppose that we have one of the following cases: $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$ or $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\pm\infty}{\pm\infty}$ where a can be any real number, infinity or negative infinity.

In these cases we have, $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$

Example 43

$$\lim_{x\to 0}\frac{\sin x}{x}=\frac{0}{0}\ I.F$$

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{(\sin x)'}{x'}$$
$$=\lim_{x\to 0}\cos x$$
$$=1$$

Example 44

$$\lim_{x\to 1}\frac{5x^4-4x^2-1}{10-x-9x^3}=\frac{5-4-1}{10-1-9}=\frac{0}{0}\ I.F$$

$$\lim_{x\to 1}\frac{5x^4-4x^2-1}{10-x-9x^3}=\lim_{x\to 1}\frac{20x^3-8x}{-1-27x^2}$$
$$=\frac{20-8}{-1-27}$$
$$=-\frac{3}{7}$$

Unit summary

1. The derivative of a function $f(x)$ with respect to x is denoted by $f'(x)$ and defined as $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ provided that the limit exists.
2. If f is a constant function, say $f(x)=c$, for all x, then $\frac{df}{dx}=\frac{d}{dx}(c)=0$

3. If $\boldsymbol{n}$ is any real number, then $\frac{d}{dx}x^n = nx^{n-1}$ for all x where the powers x^n and x^{n-1} are defined.
4. If f is a differentiable function of x, *and c is a constant, then* $\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$
5. Let $D(I,\mathbb{R})$ be the set of functions differentiable on I. If $f \in D(I,\mathbb{R})$ and $g \in D(I,\mathbb{R})$, then $f \pm g \in D(I,\mathbb{R})$. In addition $\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$
6. If f *and* g are differentiable at x, then so is their product $f \cdot g$, and $\frac{d}{dx}(f \cdot g) = \frac{df}{dx} + f\frac{dg}{dx}$.
7. Let $D(I,\mathbb{R})$ be the set of functions differentiable on I. If $f \in D(I,\mathbb{R})$ then $\frac{1}{f} \in D(I,\mathbb{R})$ $f \neq 0$. Moreover $\frac{d}{dx}\left(\frac{1}{f}\right) = -\frac{\frac{df}{dx}}{f^2}$
8. If f and g are differentiable at x and if $g(x) \neq 0$, then the quotient $\frac{f}{g}$ is differentiable at x, and $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$
9. If $f \in D(I,\mathbb{R})$ and $g \in D(I,\mathbb{R})$, then $g \circ f \in D(I,\mathbb{R})$. In addition $(g \circ f)' = g'(f)f'$
10. Chain rule: $\frac{d}{dx}\left[f\left[g(x)\right]\right] = f'\left[g(x)\right]g'(x)$
11. The equation of the **tangent line** to the curve at point (x_0, y_0), is $T \equiv y - y_0 = f'(x_0)(x - x_0)$
12. The **normal line** to the curve at point (x_0, y_0) is the perpendicular line to the tangent line of the curve at point (x_0, y_0). Its equation is of the form $N \equiv y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

13. If (x_0, y_0) are points on the graph of $y = f(x)$, then we define $m = \frac{y_1 - y_0}{x_1 - x_0}$ to be the **average rate** at which y changes with x over the interval $[x_0, x_1]$. If $y = f(x)$ and $f(x)$ is differentiable at x_0, then we define $n = \frac{dy}{dx}|_{x=x_0}$ to be the **instantaneous rate** at which y changes with x_0 at x_0.

14. We say that $x = c$ is a critical point for the function $f(x)$ if $f(c)$ exists and if either one of the following is true
    - $f'(c) = 0$ or
    - $f'(c)$ does not exist.

15. **Extreme Value Theorem:** Suppose that $f(x)$ is continuous on the interval $[a, b]$ then there are two numbers $a \le c, d \le b$ so that $f(c)$ is an absolute maximum for the function and $f(d)$ is an absolute minimum for the function.

16. **Fermat's Theorem:** If $f(x)$ has a relative extrema at $x = c$ and $f'(c)$ exists then $x = c$ is a critical point of $f(x)$. In fact, it will be a critical point such that $f'(c) = 0$.

17. If $f'(x) > 0$ for every x on some interval I, then $f(x)$ is increasing on the interval.

18. If $f'(x) < 0$ for every x on some interval I, then $f(x)$ is decreasing on the interval.

If $f'(x) = 0$ for every x on some interval I, then $f(x)$ is constant on the interval

If $f''(x)$ for all x in some interval I then $f(x)$ is concave up on I.

If $f''(x)$ for all x in some interval I then $f(x)$ is concave down on I.

19. **Rolle's Theorem:** Suppose that $f(x)$ is a function that satisfies all of the following:

- $f(x)$ is continuous on the closed interval $[a,b]$,
- $f(x)$ is differentiable on the open interval (a,b),
  $f(a)=f(b)$

Then, there is a number c such that $a<c<b$ and $f'(c)=0$.

20. **Mean Value Theorem:** Suppose that $f(x)$ is a function that satisfies both of the following.

- $f(x)$ is continuous on the closed interval $[a,b]$.
- $f(x)$ is differentiable on the open interval (a,b).

Then, there is a number c such that $a<c<b$ and $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

21. **L'Hôpital's rule**: If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{0}{0}$ or $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{\pm\infty}{\pm\infty}$ where a can be any
real number, infinity or negative infinity. Then
$\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$

Revision exercise

1. Write down the derivative of each of the following functions:
   a) x^4 b) $4x^3$ c) $8x^2$ d) x^2-4x+5
2. a) Find equation of the tangent to the curve $y=x^2-4x-5$ at the point $(1,-8)$
   b) Find the equation of the tangent to the curve $y=3x^2-4x$ at the point where $x=\dfrac{2}{3}$.
3. For the function $f(x)=3x^2-10x+3$
   a) find the value of $f'(x)$ when $f(x)=0$
   b) find the value of $f(x)$ when $f'(x)=0$

4. Write down the first and the second derivatives of each of the following:

   a) $5x-4$ b) $3x^2-6x-5$ c) $2x^3-5x^2+4x+2$ d) $x^3-\frac{2}{x}$

5. A ball is thrown vertically into the air so that it reaches a height of $y=19.6t-4.9t^2$ meters in t seconds.
   a) Find the time taken and acceleration of the ball at time t seconds.
   b) Find the time taken for the ball to reach its highest point.
   c) How high did the ball rise?
   d) At what time(s) would the ball be at half its maximum height?

6. The function $f(x)=2x^3+ax^2+bx$ has stationary points at $x=-1, x=2$. Find the values of the real numbers a and b and sketch the graph of the function.

7. Find the largest and smallest values of $f(x)=x(x-2)^2$ for

   a) $-1\le x\le 3$ b) $0\le x\le 2$ c) $-\frac{1}{3}\le x\le\frac{2}{3}$

8. Find the rate of change of the area of a square with respect to the length of its side when the side is *4ft*.

9. Find the rate of change of the volume of a sphere (given by $V=\frac{4}{3}\pi r^3$) with respect to it radius r when the radius is *2m*.

10. Find the intervals of increase and decrease of the following functions

    a) $f(x)=x^3-4x+1$ b) $f(x)=(x^2-4)^2$ c) $f(x)=x^3(5-x)^2$

In Exercises 11-14, find $\frac{dy}{dx}$ in terms of x and y:

11. $xy-x+2y=1$
12. $x^2+xy=y^3$
13. $x^2y^3=2x-y$
14. $\frac{x-y}{x+y}=\frac{x^2}{y}+1$

In Exercises 15-18, find y', y'', y''':

15. $y=(3-2x)^7$
16. $y=\frac{6}{(x-1)^2}$
17. $y=x^{\frac{1}{3}}-x^{-\frac{1}{3}}$
18. $y=(x^2+3)\sqrt{x}$

Unit 7

Vector space of real numbers

My goals

By the end of this unit, I will explain:

- Dot product and properties.
- Modulus or magnitude of vectors
- Angle between two vectors

Introduction

A vector space (also called a linear space) is a collection of objects called vectors, which may be added together and multiplied by numbers, called scalars in this context.

To put it really simply, vectors are basically all about directions and magnitudes. These are critical in basically all situations.

In physics, vectors are often used to describe forces, and forces add as vectors do.

Classical mechanics: block sliding down a ramp: you need to calculate the force of gravity (a vector down), the normal force (a vector perpendicular to the ramp), and a friction force (a vector opposite the direction of motion).

Required outcomes

After completing this unit, the learners should be able to:

- Find the norm of a vector.
- Calculate the scalar product of two vectors.
- Calculate the angle between two vectors.
- Apply and transfer the skills of vectors to other area of knowledge.

1. Euclidian vector space IR^2

A vector space (also called a linear space) is a collection of objects called vectors, which may be added together and multiplied ("scaled") by numbers, called scalars in this context. Scalars are often taken to be real numbers, but there are also vector spaces with scalar multiplication by complex numbers, rational numbers, or generally any field.

Dot product and properties and magnitude of a vector

Activity 1

Evaluate:

a) $(1,4)\cdot(3,2)$ b) $(-4,2)\cdot(1,2)$

Hint: $(a,b)\cdot(c,d) = a\cdot c + b\cdot d$

Scalar product and properties

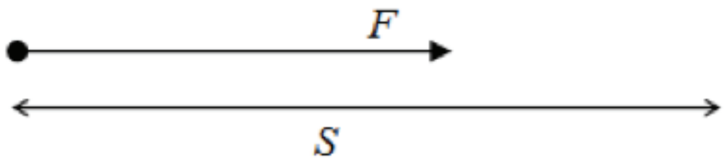
The scalar product or dot product (or sometimes inner product) is an algebraic operation that takes two coordinate vectors and returns a single number.

Algebraically, it is the sum of the products of the corresponding coordinates of the two vectors. That is, the scalar product of vectors $\vec{u} = (a_1, a_2)$ and $\vec{v} = (b_1, b_2)$ of plane is defined by $\vec{u}\cdot\vec{v} = a_1b_1 + a_2b_2$

We can illustrate this scalar product in terms of work done by a force on the body:

Suppose that a person is holding a heavy weight at rest. This person may say and feel he is doing hard work but in fact none is being done on the weight in the scientific sense. Work is done when a force moves its point of application along the direction of its line of action.

If the constant force F and the displacement S are in the same direction and we define the work W done by the force on the body by $W = F \cdot S$



Properties of scalar product

a) If $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$, then $\vec{u} \cdot \vec{v} = \vec{0}$.

b) If $\vec{u} \parallel \vec{v}$ and $\vec{u}, \vec{v}$ have same direction, then $\vec{u} \cdot \vec{v} > 0$.

c) If $\vec{u} \parallel \vec{v}$ and $\vec{u}, \vec{v}$ have opposite direction, then $\vec{u} \cdot \vec{v} < 0$.

d) If $\vec{u} \perp \vec{v}$, then $\vec{u} \cdot \vec{v} = 0$.

e) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

f) $\vec{u} \cdot \left(a\vec{v} + b\vec{w}\right) = a\vec{v} \cdot \vec{u} + b\vec{u} \cdot \vec{w}$, $\left(a\vec{u} + b\vec{v}\right) \cdot \vec{w} = a\vec{u} \cdot \vec{w} + b\vec{v} \cdot \vec{w}$.

g) $\vec{u} \cdot \vec{u} > 0$, $\vec{u} \neq \vec{0}$.

We define the square of $\vec{u}$ to be $\vec{u} \cdot \vec{u} = \left(\vec{u}\right)^2$

Example 1

Let $\vec{u}$ =(2,4) and $\vec{v}$ (-5,0). Find $\vec{u} . \vec{v}$, $(\vec{u})^2$ and $(\vec{v})^2$

Solution

The scalar product of the vector $\vec{u} = (2,4)$ and vector $\vec{v} = (-5,0)$ is $\vec{u} \cdot \vec{v} = 2(-5) + 0 = -10$

The square of the vector $\vec{u} = (10,4)$ is

$\left(\vec{u}\right)^2 = 10(10) + 4(4) = 100 + 16 = 116$

$(\vec{v})^2 = -5\ (-5) + 0 = 25$

Notice:

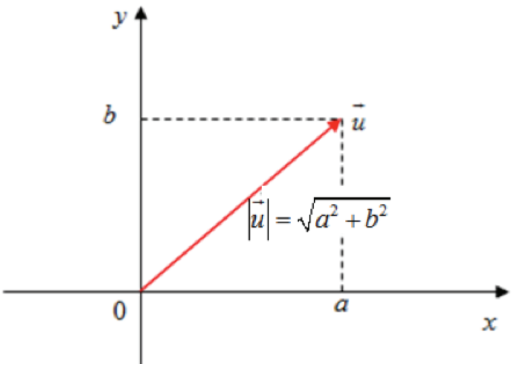
Two vectors are perpendicular if their scalar product of two vectors is zero.

Magnitude or modulus of vectors

The magnitude of the vector $\vec{u}$ noted by $\left\|\vec{u}\right\|$ is defined as its length and is the square root of its square. That is $\left\|\vec{u}\right\| = \sqrt{\left(\vec{u}\right)^2}$ or $\left\|\vec{u}\right\|^2 = \left(\vec{u}\right)^2$. Thus if $\vec{u} = (a,b)$ then $\left\|\vec{u}\right\| = \sqrt{a^2 + b^2}$

Note that the notation of absolute value $|\ |$ is also used for the magnitude of a vector.

That is the magnitude of a vector $\vec{u}$ is also denoted by $\left|\vec{u}\right|$.



Consequences

a) If $\vec{u} = \vec{0}$ then $\left\|\vec{u}\right\| = 0$

b) $\left\|k\vec{u}\right\| = \left\|k\right\|\left\|\vec{u}\right\|$, k is a real number.

Example 2

Find the norm of the vector $\vec{v} = (3,4)$.

Solution

The norm is

$\left\|\vec{v}\right\| = \sqrt{9+16} = 5$

Example 3

Find the norm of the vector $\vec{u} = (-1,4)$

Solution

The norm is

$\left\|\vec{u}\right\| = \sqrt{1+16} = \sqrt{17}$.

Exercise 1

1. Find the norm of

   a) $\vec{u}=(-3,4)$ b) $\vec{v}=(3,1)$

2. Consider the vectors $\vec{u}=(4,5)$, $\vec{v}=(-3,1)$ in plane. Find

   a) the norm of vector $\vec{u}$ and vector $\vec{v}$

   b) the scalar product of vector $\vec{u}$ and vector $\vec{v}$

Angle between two vectors

Activity 2

In xy plane,

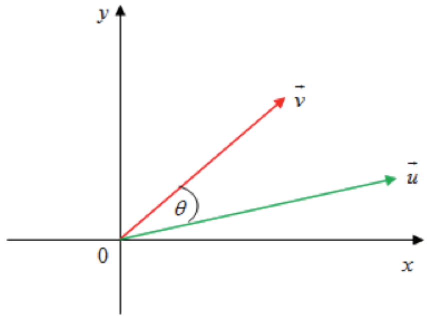
1. Draw the vector $\vec{u}=(3,0)$ with tail at $(0,0)$.
2. Draw another vector $\vec{v}$ with length $3\sqrt{2}$ such that the angle between $\vec{u}$ and vector $\vec{v}$ will be $\theta=45^{\circ}$

What can you say about the sides of the triangle formed and the angle θ ?

Geometrically, the scalar product of two vectors $\vec{u}=(a,b)$ and $\vec{v}=(c,d)$ of plane is given by $\vec{u}\cdot\vec{v}=\|\vec{u}\|\|\vec{v}\|\cos\theta$. Where θ is the angle between vectors $\vec{u}$ and $\vec{v}$. From this relation, we have $\cos\theta=\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$ or $\cos\theta=\frac{a\cdot c+b\cdot d}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}$.

We deduce that the angle θ between two vectors $\vec{u}=(a,b)$ and $\vec{v}=(c,d)$ is given by $\theta=\cos^{-1}\left(\frac{a\cdot c+b\cdot d}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}\right)$.

Where $\cos^{-1}$ denote the inverse function of cosine.

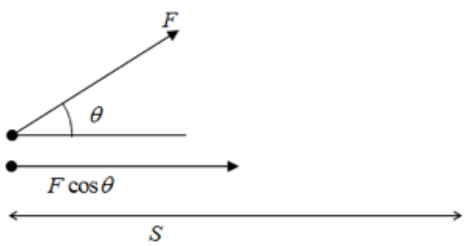


We can illustrate the scalar product in terms of work done by the force on the body:

If the constant force F and the displacement S are in the same direction, the work W done by the force on the body is $W = F \cdot S$

If the force does not act in the direction in which motion occurs but an angle to it , then the work done is defined as the product of the component of the force in the direction of motion and the displacement in that direction.

$W = FS\cos\theta$



Notice

- Two angles are perpendicular if the angle between them is a multiple of a right angle.
- Two angles are parallel and with the same direction if the angle between them is a multiple of a zero angle.
- Two angles are parallel and with the opposite direction if the angle between them is a multiple of a straight angle.

Example 4

Find the angle between vectors $\vec{u}=(3,0)$ and $\vec{v}=(5,5)$.

Solution

Let α be the angle between these two vectors.

$$\alpha=\cos^{-1}\left(\frac{3\cdot 5+0\cdot 5}{\sqrt{3^3+0^2}\sqrt{5^2+5^2}}\right)=\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$\alpha=45^\circ$$

Example 5

Calculate the dot product and the angle formed by the following vectors:

$\vec{u}=(3,4)$ and $\vec{v}=(-8,6)$

Solution

$$\vec{u}\cdot\vec{v}=3\cdot(-8)+4\cdot 6=0$$

$$\alpha=\cos^{-1}\left(\frac{0}{\sqrt{3^2+4^2}\sqrt{(-8)^2+6^2}}\right)=\cos^{-1}(0)$$

$$\alpha=90^\circ$$

Example 6

Calculate the angles of the triangle with vertices: $A=(6,0), B=(3,5)$ and $C=(-1-1)$.

Solution

$\overrightarrow{AB}=(-3,5)$, $\overrightarrow{BA}=(3,-5)$, $\overrightarrow{AC}=(-7,-1)$,

$\overrightarrow{CA}=(7,1)$, $\overrightarrow{BC}=(-4,-6)$, $\overrightarrow{CB}=(4,6)$

$$\angle A=\cos^{-1}\left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\|\overrightarrow{AB}\|\|\overrightarrow{AC}\|}\right)$$

$$=\cos^{-1}\frac{(-3)(-7)+5(-1)}{\sqrt{(-3)^2+5^2}\sqrt{(-7)^2+(-1)^2}}$$

$$=\cos^{-1}\frac{16}{\sqrt{34}\sqrt{50}}$$

$$\angle A\approx 67.2^\circ$$

$$\angle B=\cos^{-1}\left(\frac{\overrightarrow{BC}\cdot\overrightarrow{BA}}{\|\overrightarrow{BC}\|\|\overrightarrow{BA}\|}\right)$$

$$=\cos^{-1}\frac{18}{\sqrt{52}\sqrt{34}}$$

$$\angle B\approx 64.6^\circ$$

$$\angle C = \cos^{-1}\left(\frac{\overrightarrow{CA}\cdot\overrightarrow{CB}}{\left\|\overrightarrow{CA}\right\|\left\|\overrightarrow{CB}\right\|}\right)$$

$$= \cos^{-1}\frac{34}{\sqrt{50}\sqrt{52}}$$

$$\angle C \approx 48.2^{\circ}$$

Example 7

Find the value of k if the angle between $\vec{u} = (k,3)$ and $\vec{v} = (4,0)$ is 45°

Solution

$$\cos 45^{0} = \frac{4k}{\sqrt{k^2+9}\sqrt{16}}$$

$$\frac{\sqrt{2}}{2} = \frac{4k}{4\sqrt{k^2+9}}$$

$$\frac{\sqrt{2}}{2} = \frac{k}{\sqrt{k^2+9}} \Leftrightarrow 2k = \sqrt{2}\sqrt{k^2+9}$$

$$4k^2 = 2k^2 + 18$$

$$k^2 = 9 \Rightarrow k = \pm 3$$

The value of k is 3 since $\cos 45^{0}$ must be positive.

Exercise 2

1. Find the angle formed by vectors;

   a) $\vec{u} = (5,6),\ \vec{v} = (-1,4)$ b) $\vec{u} = (3,5),\ \vec{v} = (-1,6)$

2. Given the vectors $\vec{u} = (2,k)$ and $\vec{v} = (3,-2)$, calculate the value of k so that the vectors $\vec{u}$ and $\vec{v}$ are:

   a) perpendicular.

   b) parallel.

   c) make an angle of 60°.

Application

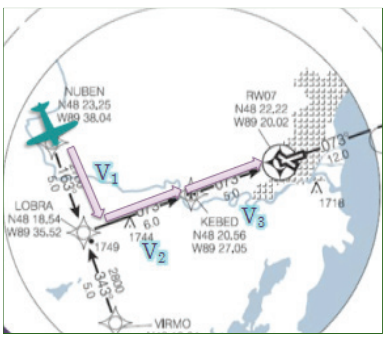
In physics,

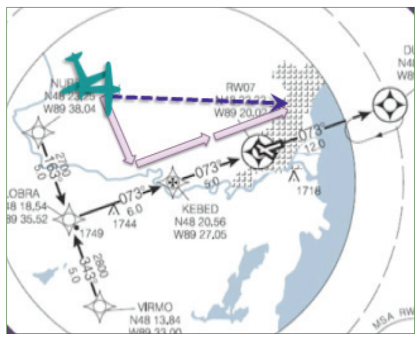
Vectors are fundamental in the physical sciences. They can be used to represent any quantity that has magnitude, has direction, and which adheres to the rules of vector addition. An example is velocity, the magnitude of which is speed.

For example, the velocity 7 meters per second upward could be represented by the vector (0,7). Another quantity represented by a vector is force, since it has a magnitude and direction and follows the rules of vector addition. Vectors also describe many other physical quantities, such as linear displacement, displacement, linear acceleration, angular acceleration, linear momentum, and angular momentum. Other physical vectors, such as the electric and magnetic field, are represented as a system of vectors at each point of a physical space; that is, a vector field.

In geography,

Vectors can be used in air plane navigation





Unit summary

1. The vector joining point $A(a_1,a_2)$ and $B(b_1,b_2)$ is given by $\overrightarrow{AB}=(b_1-a_1,b_2-a_2)$.

2. The scalar product of vectors $\vec{u}=(a_1,a_2)$ and $\vec{v}=(b_1,b_2)$ of plane is defined by $\vec{u}\cdot\vec{v}=a_1b_1+a_2b_2$

3. The magnitude of the vector $\vec{u}$ noted by $\|\vec{u}\|$ is defined as $\|\vec{u}\|=\sqrt{a^2+b^2}$

4. The angle θ between two vectors $\vec{u}=(a,b)$ and $\vec{v}=(c,d)$ is

given by $\theta = \cos^{-1}\left(\dfrac{a \cdot c + b \cdot d}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}\right)$. Where $\cos^{-1}$ denote the inverse function of cosine.

Revision exercise

1. Find the magnitude of the vectors

   a) $\vec{u} = (12,3)$

   b) $\vec{v} = (-9,4)$

   c) $\vec{w} = (6,-10)$

2. Find the scalar product of vectors

   a) $\vec{u} = (1,4)$ and $\vec{v} = (5,6)$

   b) $\vec{u} = (-10,14)$ and $\vec{v} = (2,0)$

   c) $\vec{u} = (12,-4)$ and $\vec{v} = (3,8)$

3. Find the cosine of the angle between the vectors (2, 5) and (−1, 3) as well as the angle itself (in radians and degrees).

4. Find the angle between

   a) $\vec{a} = (3,4)$ and $\vec{b} = (4,3)$

   a) $\vec{a} = (7,1)$ and $\vec{b} = (5,5)$

5. Find the value of k if the angle between and is:

   a) 90°
   b) 0°
   c) 45°

Unit 8

Matrices and determinants of order 2

My goals

By the end of this unit, I will explain:

- Matrices of order two.
- Determinants of matrices of order 2.
- Inverse of matrix.

Introduction

A matrix is a rectangular arrangement of numbers, expressions, symbols which are arranged in rows and columns.

Matrices play a vital role in the projection of a three dimensional image into a two dimensional image. Matrices and their inverse are used by programmers for coding or encrypting a message. Matrices are applied in the study of electrical circuits, quantum mechanics and optics. A message is made as a sequence of numbers in a binary format for communication and it follows code theory for solving. Hence with the help of matrices, those equations are solved. Matrices are used for taking seismic surveys.

Required outcomes

After completing this unit, the learners should be able to:

- Define matrices.
- Perform operations on matrices of order 2.
- Determine determinant of matrix.
- Determine the inverse of a matrix of order 2.

1. Square Matrices of order two

Activity 1

A shop sold 20 cell phones and 31 computers in a particular month. Another shop sold 45 cell phones and 23 computers in the same month. Present this information as an array of rows and columns.

A matrix is every set of numbers or terms arranged in a rectangular shape, forming rows and columns. In square matrix of order two, the number of rows is equal to the number of columns equal to 2 and it has the following form;

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

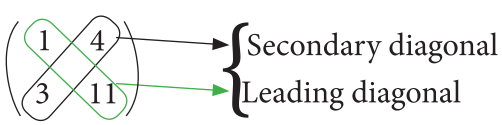
The elements a_{11} and a_{22} constitute the **principal diagonal** (or **leading diagonal** or **main diagonal** or **major diagonal**

or **primary diagonal**) elements a_{12} and a_{21} constitute the **secondary diagonal** (or **minor diagonal** or **antidiagonal** or **counterdiagonal**)

The subscript of a_{ij} means that a is an element in row i and column j.

Example 1

The following matrix is a square matrix of order two



Notice:

- The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is said to be the identity (or unit) matrix.

- The matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is said to be the zero (or null) matrix.

Notice: Equality of matrices

Two matrices are equal if the elements of the two matrices that occupy the same position are equal.

If $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then $\begin{cases} a_{11} = b_{11} \\ a_{21} = b_{21} \\ a_{12} = b_{12} \\ a_{22} = b_{22} \end{cases}$

Example 2

If $A = \begin{pmatrix} 3y+2 & 2 \\ 2x+1 & 1 \end{pmatrix}$ and

$B = \begin{pmatrix} y-3 & 2 \\ 5 & 6 \end{pmatrix}$

are equal. Find the value of x and y

Solution

$$3y+2 = y-3 \Rightarrow y = -\frac{5}{2}$$

$$2x+1 = 5 \Rightarrow x = 2$$

Exercise 1

Give five examples of matrices of order two.

Operations on matrices

Activity 2

Consider the matrices $A = \begin{pmatrix} 13 & 4 \\ 6 & 10 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 10 \\ 3 & 4 \end{pmatrix}$, find;

1. $A+3B$
2. $2A-B$
3. Interchange the rows and column of matrix A and matrix B.

Adding matrices

Given two matrices, $A = (a_{ij})$ and $B = (b_{ij})$, the matrix sum is defined as: $A + B = (a_{ij} + b_{ij})$. That is, the resultant matrix's elements are obtained by adding the elements of the two matrices that occupy the same position.

If $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then

$$A+B=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} \quad \text{and}$$

$$A-B=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}-\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}=\begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} \\ a_{21}-b_{21} & a_{22}-b_{22} \end{pmatrix}$$

Example 3

If $A=\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix}$, find the sum and the difference

Solution

$$A+B=\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}+\begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix}=\begin{pmatrix} 4 & 4 \\ 3 & 7 \end{pmatrix}$$

$$A-B=\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix}=\begin{pmatrix} 2 & 0 \\ -5 & -5 \end{pmatrix}$$

Properties

1. **Closure**

   The sum of two matrices of order two is another matrix of order two.

2. **Associative**

   $A+(B+C)=(A+B)+C$

3. **Additive identity**

   $A+0=A$, Where **0** is the zero-matrix.

4. **Additive inverse**

   $A+(-A)=0$

The opposite matrix has each of its elements change sign.

5. **Commutative**

$A + B = B + A$

Scalar multiplication

Given a matrix, $A = \left(a_{ij}\right)$, and a real number, $k \in IR$, the product of a real number by a matrix is $k \cdot A = \left(k\, a_{ij}\right)$

If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\alpha \in \mathbb{R}$, then

Example 4

If $A = \begin{pmatrix} -3 & 6 \\ 5 & 2 \end{pmatrix}$,

find $2A$

Solution

$$2A = 2\begin{pmatrix} -3 & 6 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} -6 & 12 \\ 10 & 4 \end{pmatrix}$$

Properties

Let a and b be two matrices of order two and α, β be any real numbers.

1. $\alpha(\beta A) = (\alpha\beta)A$
2. $\alpha(A + B) = \alpha A + \alpha B$
3. $(\alpha + \beta)A = \alpha A + \beta A$
4. $1A = A$

Multiplying matrices

Two matrices a and b of order two can be multiplied together. The element of the product matrix is obtained by multiplying every element in row ***i*** of matrix a by each element of column ***j*** of matrix b and then adding them together.

If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then

$$A \cdot B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

Example 5

If $A=\begin{pmatrix}1 & 3\\ 2 & 5\end{pmatrix}$ and $B=\begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}$, find the product AB

Solution

$$A\cdot B=\begin{pmatrix}1 & 3\\ 2 & 5\end{pmatrix}\cdot\begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}$$
$$=\begin{pmatrix}1\cdot 2+3\cdot 1 & 1\cdot 0+3\cdot 1\\ 2\cdot 2+5\cdot 1 & 2\cdot 0+5\cdot 1\end{pmatrix}$$
$$=\begin{pmatrix}5 & 3\\ 9 & 5\end{pmatrix}$$

Properties

1. **Associative**

   $A\cdot(B\cdot C) = (A\cdot B)\cdot C$

2. **Multiplicative identity**

   $A\cdot I = A$, Where $\boldsymbol{I}$ is the identity or unit matrix.

3. **Not commutative**

   $A\cdot B \neq B\cdot A$

4. **Distributive**

   $A\cdot(B + C) = A\cdot B + A\cdot C$

Example 6

Given the matrices:

$$A=\begin{pmatrix}4 & 5\\ 2 & 1\end{pmatrix} \text{ And } B=\begin{pmatrix}2 & 3\\ 1 & 0\end{pmatrix}$$

$$A\cdot B=\begin{pmatrix}4 & 5\\ 2 & 1\end{pmatrix}\cdot\begin{pmatrix}2 & 3\\ 1 & 0\end{pmatrix}$$
$$=\begin{pmatrix}4\cdot 2+5\cdot 1 & 4\cdot 3+5\cdot 0\\ 2\cdot 2+1\cdot 1 & 2\cdot 3+1\cdot 0\end{pmatrix}$$
$$=\begin{pmatrix}13 & 12\\ 5 & 6\end{pmatrix}$$

$$B\cdot A=\begin{pmatrix}2 & 3\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}4 & 5\\ 2 & 1\end{pmatrix}$$
$$=\begin{pmatrix}2\cdot 4+3\cdot 2 & 2\cdot 5+3\cdot 1\\ 1\cdot 4+0\cdot 2 & 1\cdot 5+0\cdot 1\end{pmatrix}$$
$$=\begin{pmatrix}14 & 13\\ 4 & 5\end{pmatrix}$$

Notice:

- If $AB=0$, it does not necessarily follow that $A=0$ or $B=0$.

Example 7

$$A=\begin{pmatrix}1 & 2\\ -2 & -4\end{pmatrix}\neq 0,\quad B=\begin{pmatrix}2 & -1\\ -1 & \frac{1}{2}\end{pmatrix}\neq 0 \text{ But}$$

$$AB=\begin{pmatrix}1 & 2\\ -2 & -4\end{pmatrix}\begin{pmatrix}2 & -1\\ -1 & \frac{1}{2}\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

Commuting matrices

In general the multiplication of matrices is not commutative, i.E, $AB\neq BA$, but we can have the case where two matrices A and B satisfy $AB=BA$. In this case A and B are said to be **commuting**.

Example 8

The matrices $A=\begin{pmatrix}5 & 9\\ 0 & 2\end{pmatrix}$, $B=\begin{pmatrix}2 & 3\\ 0 & 1\end{pmatrix}$ commute as,

$$\left.\begin{array}{l} AB=\begin{pmatrix}5 & 9\\ 0 & 2\end{pmatrix}\begin{pmatrix}2 & 3\\ 0 & 1\end{pmatrix}=\begin{pmatrix}10 & 24\\ 0 & 2\end{pmatrix}\\ BA=\begin{pmatrix}2 & 3\\ 0 & 1\end{pmatrix}\begin{pmatrix}5 & 9\\ 0 & 2\end{pmatrix}=\begin{pmatrix}10 & 24\\ 0 & 2\end{pmatrix}\end{array}\right\| \Rightarrow AB=BA$$

Transpose of matrix

Given matrix A, we define the transpose of matrix A, noted A^t, to be another matrix where the elements in the columns and rows have interchanged. In other words, the rows become the columns and the columns become the rows. the transpose of $A=\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix}$ is $A^t=\begin{pmatrix}2 & 3\\ 1 & 4\end{pmatrix}$. For example

Example 9

Find the transpose of $A=\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix}$

Solution

The transpose is $A^t=\begin{pmatrix}2 & 3\\ 1 & 4\end{pmatrix}$.

Properties of transpose of matrices

a) $\left(A^t\right)^t = A$

b) $\left(A+B\right)^t = A^t + B^t$

c) $\left(\alpha \cdot A\right)^t = \alpha \cdot A^t$

d) $\left(A \cdot B\right)^t = B^t \cdot A^t$

Exercise 2

1. If $A=\begin{pmatrix}-1 & -3\\ 1 & 1\end{pmatrix}$ and $B=\begin{pmatrix}6 & 3\\ 2 & 1\end{pmatrix}$, find:

   a) $A+B$ b) $3A-B$ c) AB d) A^2 e) B^3

2. Given $A=\begin{pmatrix}x+1 & -3\\ 4z+x & 3y+4\end{pmatrix}$ and $B=\begin{pmatrix}2x-6 & -3\\ 1 & 1\end{pmatrix}$

   If $A=B$ find the value of x, y and z and hence find:

   a) A b) A^t

2. Determinants and inverse of matrices

Activity 3

Find:

1. $\begin{vmatrix}1 & 0\\ 2 & 1\end{vmatrix}$ 2. $\begin{vmatrix}-2 & -4\\ 3 & 6\end{vmatrix}$ 3. $\begin{vmatrix}3 & 1\\ 6 & 8\end{vmatrix}$ 4. $\begin{vmatrix}12 & 3\\ -2 & 9\end{vmatrix}$

Hint: $\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

Determinants of matrices

Every square matrix, A of order two is assigned a particular scalar quantity called the **determinant of A**, denoted by $|A|$ or by $det(A)$.

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

This determinant is calculated as follows:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example 10

$$\begin{vmatrix} 3 & 5 \\ 2 & -1 \end{vmatrix} = 3(-1) - 2(5) = -3 - 10 = -13$$

Properties of the determinants of matrices

1. The determinant of matrix a and its transpose a^t are equal.

   $|A^t| = |A|$

Example 11

Let $A = \begin{pmatrix} -7 & 8 \\ 9 & -1 \end{pmatrix}$ then $A^t = \begin{pmatrix} -7 & 9 \\ 8 & -1 \end{pmatrix}$ and $|A| = |A^t| = -65$

2. $|A| = 0$ if:

- It has two equal lines

Example 12

Let $A = \begin{pmatrix} 15 & -5 \\ 15 & -5 \end{pmatrix}$ then $|A| = \begin{vmatrix} 15 & -5 \\ 15 & -5 \end{vmatrix} = 0$

Since the first and second rows are equal.

- All elements of a line are zero.

Example 13

Let $A=\begin{pmatrix}-8 & 6\\ 0 & 0\end{pmatrix}$ then $|A|=\begin{vmatrix}-8 & 6\\ 0 & 0\end{vmatrix}=0$

Since all elements of the second row are zero.

- The elements of a line are a linear combination of the other.

Example 14

Let $A=\begin{pmatrix}3 & 2\\ 6 & 4\end{pmatrix}$ then $|A|=\begin{vmatrix}3 & 2\\ 6 & 4\end{vmatrix}=0$

Since the second row is a multiple of the first row.

3. If a determinant switches two parallel rows or columns, its determinant changes sign.

Example 15

Let $A=\begin{pmatrix}2 & 3\\ 1 & 2\end{pmatrix}$, then $|A|=\begin{vmatrix}2 & 3\\ 1 & 2\end{vmatrix}=-\begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix}$

Since the two rows have been switched

4. If a determinant is multiplied by a real number, any row or column can be multiplied by the above mentioned number, but only one.

Example 16

Let $A=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$, we can find $2|A|$ as follows:

$$2|A|=2\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}=2\begin{vmatrix}2\cdot 2 & 1\cdot 2\\ 1 & 2\end{vmatrix}=\begin{vmatrix}4 & 2\\ 1 & 2\end{vmatrix}=8-2=6$$

$$\text{or } 2|A|=2\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}=2\begin{vmatrix}2\cdot 2 & 1\\ 1\cdot 2 & 2\end{vmatrix}=\begin{vmatrix}4 & 1\\ 2 & 2\end{vmatrix}=8-2=6$$

$$\text{or } 2|A|=2\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}=2\begin{vmatrix}2 & 1\cdot 2\\ 1 & 2\cdot 2\end{vmatrix}=\begin{vmatrix}2 & 2\\ 1 & 4\end{vmatrix}=8-2=6$$

$$\text{or } 2|A|=2\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}=2\begin{vmatrix}2 & 1\\ 1\cdot 2 & 2\cdot 2\end{vmatrix}=\begin{vmatrix}2 & 1\\ 2 & 4\end{vmatrix}=8-2=6$$

5. The determinant of a product equals the product of the determinants.

$$|A \cdot B| = |A| \cdot |B|$$

Example 17

Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 6 \\ 0 & 4 \end{pmatrix}$

$$A \cdot B = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 14 \\ 3 & 10 \end{pmatrix}$$

$$|A \cdot B| = \begin{vmatrix} 3 & 14 \\ 3 & 10 \end{vmatrix} = 30 - 42 = -12$$

But $|A| = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$, $|B| = \begin{vmatrix} 3 & 6 \\ 0 & 4 \end{vmatrix} = 12$ and $|A||B| = -1 \times 12 = -12$

Exercise 3

If $A = \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix}$, find;

1. $|A|$ 2. $|B|$ 3. $|AB|$ 4. $|A^t|$

Matrix inverse

Activity 4

For the following matrix, interchange the elements of the leading diagonal; change the signs of the elements of the other diagonal, divide by the determinant of the matrix and give the result

$$A = \begin{pmatrix} 10 & 2 \\ 6 & 3 \end{pmatrix}$$

Multiply your result with the orginal (the given) matrix. What is your observation from that product?

Calculating matrix inverse of matrix A, is to find matrix A^{-1} such that,

$A \cdot A^{-1} = A^{-1} \cdot A = I$, Where I is the identity matrix.

Consider the following matrix

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

The inverse of a is $A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

If $\det A = 0$ (i.e the determinant is zero) the matrix has no inverse and is said to be a **singular** matrix.

Example 18

Find the inverse of

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

Solution

$\det A = 1 - 0 = 1$

$$A^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

Example 19

Find the inverse of

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$

Solution

$\det A = 6 - 6 = 0$

Since the determinant is zero, the given matrix has no inverse.

Properties of the inverse matrix

1. $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$

2. $\left(A^{-1}\right)^{-1} = A$

3. $(\alpha \cdot A)^{-1} = \alpha^{-1} \cdot A^{-1}$

4. $\left(A^{t}\right)^{-1} = \left(A^{-1}\right)^{t}$

Exercise 4

If $A = \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix}$, find;

1. A^{-1}
2. B^{-1}
3. $(AB)^{-1}$
4. $\left(A^{t}\right)^{-1}$
5. $(4B)^{-1}$

Solving simultaneous equations

Activity 5

Express the following simultaneous equations in matrix form

1. $\begin{cases} 3x+y=9 \\ x-y=0 \end{cases}$ 2. $\begin{cases} x-y=19 \\ x+y=10 \end{cases}$

Consider the following system

$$\begin{cases} ax+by=c \\ a'x+b'y=c' \end{cases}$$

To solve this system by matrix method, first, we express the equations in matrix form as follows:

$$\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} c \\ c' \end{pmatrix}$$

We multiply both sides by the inverse of $\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$ and then

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}^{-1}\begin{pmatrix} c \\ c' \end{pmatrix}.$$

Example 20

Solve the simultaneous equation $\begin{cases} 4x-y=1 \\ -2x+3y=12 \end{cases}$ by matrix method.

Solution

$$\begin{cases} 4x-y=1 \\ -2x+3y=12 \end{cases} \Leftrightarrow \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ 12 \end{pmatrix}$$

Inverse of $\begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}$ is $\frac{1}{12-2}\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}=\frac{1}{10}\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$

Then we have

$$\begin{pmatrix} x \\ y \end{pmatrix}=\frac{1}{10}\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 12 \end{pmatrix}=\frac{1}{10}\begin{pmatrix} 15 \\ 50 \end{pmatrix}=\begin{pmatrix} 1.5 \\ 5 \end{pmatrix}$$

Thus, $x=1.5$ and $y=5$

Exercise 5

Solve the following simultaneous equation by matrix method;

1. $\begin{cases} 2x - y = 2 \\ x + 3y = 15 \end{cases}$

2. $\begin{cases} x + y = 4 \\ -x - y = -4 \end{cases}$

3. $\begin{cases} 3x + 3y = 6 \\ x - y = 0 \end{cases}$

4. $\begin{cases} 3x - y = 6 \\ 2x + 4y = 4 \end{cases}$

Unit summary

1. A matrix is every set of numbers or terms arranged in a rectangular shape, forming rows and columns.
2. Two matrices are equal if the elements of the two matrices that occupy the same position are equal.
3. Given two matrices, $A = (a_{ij})$ and $B = (b_{ij})$, the matrix sum is defined as: $A + B = (a_{ij} + b_{ij})$.
4. Two matrices A and B of order two can be multiplied together. The element of the product matrix is obtained by multiplying every element in row ***i*** of matrix A by each element of column ***j*** of matrix B and then adding them together.
5. Given matrix A, we define the transpose of matrix A, noted A^t, to be another matrix where the elements in the columns and rows have switched.
6. The determinant is of order two

   $$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

7. The inverse of A= $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

   If $\det A$ (i.e the determinant is zero) the matrix has no inverse and is said to be a **singular** matrix.

8. Consider the following system

$$\begin{cases} ax+by=c \\ a'x+b'y=c' \end{cases}$$

To solve this system by matrix method we set

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ a' & b' \end{pmatrix}^{-1} \begin{pmatrix} c \\ c' \end{pmatrix}$$

Revision exercise

1. Find the value of x and y in each of the following matrix equations;

   a) $\begin{pmatrix} 3 & -5 \\ 2 & x \end{pmatrix} + \begin{pmatrix} 1 & y \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 5 & -2 \end{pmatrix}$

   b) $\begin{pmatrix} 3 & x \\ 5 & 4 \end{pmatrix} \begin{pmatrix} 1 & y \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 13 & 7 \end{pmatrix}$

   c) $\begin{pmatrix} 2 & 1 \\ x & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 3 & y \end{pmatrix}$

2. If $A = \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix}$, find the values of m and n given that $A^2 = mA + nI$ where

   I is the identity matrix.

3. Find the possible values x can make given that $A = \begin{pmatrix} x^2 & 3 \\ 1 & 3x \end{pmatrix}$, $B = \begin{pmatrix} 3 & 6 \\ 2 & x \end{pmatrix}$

   and $AB = BA$

4. Solve the following simultaneous equations by matrix method:

   a) $\begin{cases} x-y=5 \\ 3x+2y=5 \end{cases}$

   b) $\begin{cases} x-3y=3 \\ 5x-9y=11 \end{cases}$

   c) $\begin{cases} x+3y=1 \\ 2x-4y=1 \end{cases}$

Unit 9

Measures of dispersion

My goals

By the end of this unit, I will explain:

- Variance.
- Standard deviation (including combined set of data).
- Coefficient of variation.
- Applications.

Introduction

The word dispersion has a technical meaning in statistics. The average measures the center of the data. It is one aspect of observations. Another feature of the observations is how the observations are spread about the center. The observation may be close to the center or they may be spread away from the center. If the observation are close to the center (usually the arithmetic mean or median), we say that dispersion, scatter or variation is small. If the observations are spread away from the center, we say dispersion is large.

The study of dispersion is very important in statistical data. If in a certain factory there is consistence in the wages of workers, the workers will be satisfied. But if some workers have high wages and some have low wages, there will be unrest among the low paid workers and they might go on strikes and arrange demonstrations. If in a certain country some people are very poor and some are very high rich, we say there is economic disparity. It means that dispersion is large.

Required outcomes

After completing this unit, the learners should be able to:

- Determine the measures of dispersion of a given statistical series.
- Apply and explain the standard deviation as the more convenient measure of the variability in the interpretation of data.
- Express the coefficient of variation as a measure of the spread of a set of data as a proportion of its mean.

1. Variance

Activity 1

Complete the following table if $\bar{x} = 16.875$

x	f	$x-\bar{x}$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
12	4			
13	2			
15	1			
19	4			
21	5			
	$\sum f =$			$\sum f(x-\bar{x})^2 =$

Variance measures how far a set of numbers is spread out. A variance of zero indicates that all the values are identical. Variance is always non-negative: a small variance indicates that the data points tend to be very close to the mean and hence to each other, while a high variance indicates that the data points are very spread out around the mean and from each other.

The variance is denoted and defined by:

$$\sigma^2 = \frac{\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2}{n}$$

Developing this formula, we have

$$\sigma^2 = \frac{\sum_{i=1}^{n}\left(x_i^2 - 2x_i\overline{x} + \left(\overline{x}\right)^2\right)}{n}$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \frac{1}{n}2\overline{x}\sum_{i=1}^{n} x_i + \frac{1}{n}\left(\overline{x}\right)^2\sum_{i=1}^{n} 1$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - 2\overline{x}\overline{x} + \left(\overline{x}\right)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\overline{x}\right)^2$$

Thus, the variance is also defined by:

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\overline{x}\right)^2$$

Recall that the mean of the set of n values $x_1, x_2, x_3, ..., x_n$ is denoted and defined by:

$$\overline{x} = \frac{x_1 + x_2 + x_3 + ... + x_n}{n}$$

$$= \sum_{i=1}^{n} \frac{x_i}{n}$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i$$

Example 1

Calculate the variance of the following distribution: 9, 3, 8, 8, 9, 8, 9, 18

Solution

$$\bar{x} = \frac{9+3+8+8+9+8+9+18}{8} = 9$$

$$\sigma^2 = \frac{(9-9)^2+(3-9)^2+(8-9)^2+(8-9)^2+(9-9)^2+(8-9)^2+(9-9)^2+(18-9)^2}{8} = 15$$

Example 2

Calculate the variance of the distribution of the following grouped data:

Class	[10,20[	[20,30[	[30,40[	[40,50[	[50,60[	[60,70[	[70,80[
Frequency	1	8	10	9	8	4	2

Solution

Class	x	f	xf	x^2	$x^2 f$
[10,20[	15	1	15	225	225
[20,30[	25	8	200	625	5000
[30,40[	35	10	350	1225	12250
[40,50[	45	9	405	2025	18225
[50,60[	55	8	440	3025	24200
[60,70[	65	4	260	4225	16900
[70,80[	75	2	150	5625	11250
		$\sum x = 42$	$\sum xf = 1820$	$\sum x_i^2 = 16975$	$\sum x^2 f = 88050$

$$\bar{x} = \frac{1820}{42} = 43.33$$

$$\sigma^2 = \frac{88050}{42} - (43.33)^2 = 218.94$$

Exercise 1

Find the variance of the following set of data:

1. 1,3,2,1,2,5,4,0,2,6
2. 3,2,1,5,4,6,0,4,7,8
3. 1,5,6,7,6,4,2,6,3
4. 5,4,5,5,4,5,4,4,5,3
5. 8,7,6,8,6,5,6,4,1

2. Standard deviation

Activity 2

Complete the following table

x	f	x^2	fx	fx^2
3	2			
4	3			
5	5			
7	1			
9	6			
	$\sum f =$		$\sum fx =$	$\sum fx^2 =$

The standard deviation has the same dimension as the data, and hence is comparable to deviations from the mean. We define the **standard deviation** to be the square root of the variance.

Thus, the standard deviation is denoted and defined by;

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n}} \text{ or } \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\bar{x}\right)^2}$$

The above results are used for the grouped data where x_i is the mid-interval value for the i^{th} group.

The following results follow directly from the definitions of mean and standard deviation:

- When all the data values are multiplied by a constant a, the new mean and new standard deviation are equal to a times the original mean and standard deviation. That is, the mean of $ax_1, ax_2, ax_3, ..., ax_n$ is $a(\bar{x})$ and the standard deviation is $a\sigma$.
- When a constant value, b, is added to all data values, then new mean is increased by b) However standard deviation does not change. That is, the mean of $ax_1, ax_2, ax_3, ..., ax_n$ is $\bar{x}+b$ and the standard deviation is σ.

Example 3

The six runners in a 200 meter race clocked times (in seconds) of 24.2, 23.7, 25.0, 23.7, 24.0, 24.6

a) Find the mean and standard deviation of these times.

b) These readings were found to be 10% too low due to faulty timekeeping. Write down the new mean and standard deviation.

Solution

a) $\bar{x} = \frac{24.2+23.7+25.0+23.7+24.0+24.6}{6} = 24.2 \text{ seconds}$

$$\sigma = \sqrt{\frac{(24.2-24.2)^2+(23.7-24.2)^2+(25.0-24.2)^2+(23.7-24.2)^2+(24.0-24.2)^2+(24.6-24.2)^2}{6}}$$

$= 0.473 \text{ seconds}$

b) We must divide each term 0.9 to find the correct time. The new mean is $\bar{x} = \frac{24.2}{0.9} = 26.9 \text{ sec}$. The new standard deviation is $\sigma = \frac{0.4726}{0.9} = 0.525 \text{ sec}$

The method which uses the formula for the standard deviation is not necessarily the most efficient. Consider the following:

$$\text{variance} = \frac{\sum\left(x-\bar{x}\right)^2}{n}$$

$$= \frac{\sum\left(x^2 - 2x\bar{x} + \left(\bar{x}\right)^2\right)}{n}$$

$$= \frac{\sum x^2}{n} - \frac{\sum 2x\bar{x}}{n} + \frac{\sum\left(\bar{x}\right)^2}{n}$$

$$= \frac{\sum x^2}{n} - 2\bar{x}\frac{\sum x}{n} + \left(\bar{x}\right)^2\frac{\sum 1}{n} \quad \left(\text{since } \bar{x} \text{ is a constant}\right)$$

$$= \frac{1}{n}\sum x^2 - 2\left(\bar{x}\right)^2 + \left(\bar{x}\right)^2$$

$$= \frac{1}{n}\sum x^2 - \left(\bar{x}\right)^2$$

Example 4

The heights (in meters) of six children are 1.42, 1.35, 1.37, 1.50, 1.38 and 1.30. Calculate the mean height and the standard deviation of the heights.

Solution

$$\text{Mean} = \frac{1}{6}\left(1.42 + 1.35 + 1.37 + 1.50 + 1.38 + 1.30\right) = 1.39 \ m$$

$$\text{Variance} = \frac{1}{6}\left(1.42^2 + 1.35^2 + 1.37^2 + 1.50^2 + 1.38^2 + 1.30^2\right) - 1.39^2$$

$$= 0.00386 \ m^2$$

$$\text{Standard deviation} = \sqrt{0.00386 \ m^2} = 0.0621 \ m$$

Example 5

The number of customers served lunch in a restaurant over a period of 60 days is as follows:

Number of customers served lunch	Number of days in the 60-day period
20-29	6
30-39	12
40-49	16
50-59	14
60-69	8
70-79	4

Find the mean and standard deviation of the number of customers served lunch using this grouped data.

Solution

We need the mid-interval values for all groups

Groups	Mid-interval values (x_i)	Frequency (f_i)	$f_i x_i$	$f_i x_i^2$
20-29	24.5	6	147	3601.5
30-39	34.5	12	414	14283.0
40-49	44.5	16	712	31684.0
50-59	54.5	14	763	41583.5
60-69	64.5	8	516	33282.0
70-79	74.5	4	298	22201.0
		$\sum = 60$	$\sum = 2850$	$\sum = 146635$

The mean is $\bar{x} = \frac{2850}{60} = 47.5$

The standard deviation is $\sigma = \sqrt{\frac{146635}{60} - 47.5^2} = 13.7$

Exercise 2

Find the standard deviation of the following set of data

1. 202,205,207,203,205,206,207,209
2. 1009,1011,1008,1007,1012,1010,106

3. 154,158,157,156,155,154,159
4. 7804,7806,7805,7807,7808
5. 56,54,55,59,58,57,55

3. Coefficient of variation

Activity 3

Complete the following table

x	f	x^2	fx	fx^2
10	10			
14	2			
16	14			
18	8			
20	6			
	$\sum f =$		$\sum fx =$	$\sum fx^2 =$

The coefficient of variation measures variability in relation to the mean (or average) and is used to compare the relative dispersion in one type of data with the relative dispersion in another type of data. It allows us to compare the dispersions of two different distributions if their means are positive. The greater dispersion corresponds to the value of the coefficient of greater variation.

The coefficient of variation is a calculation built on other calculations: the standard deviation and the mean as follows:

$$Cv = \frac{\sigma}{\bar{x}} \times 100$$

Where:

- σ is the standard deviation.
- $\bar{x}$ is the mean.

Example 6

One data series has a mean of 140 and standard deviation 28.28. The second data series has a mean of 150 and standard deviation 24. Which of the two has a greater dispersion?

Solution

$$Cv_1 = \frac{28.28}{140} \times 100 = 20.2\%$$

$$Cv_2 = \frac{24}{150} \times 100 = 16\%$$

The first data series has a higher dispersion.

Exercise 3

Find the coefficient of variation of the following set of data

1. 2,9,8,4,7,3,2
2. 12,11,9,8,6,10,7,9
3. 5,9,8,6,0,10,8,3,14
4. 8,10,7,11,6,12,9
5. 7,6,0,9,6,12,12,9,8,6

4. Applications

A large standard deviation indicates that the data points can spread far from the mean and a small standard deviation indicates that they are clustered closely around the mean.

Standard deviation is often used to compare real-world data against a model to test the model.

Example 7

In industrial applications, the weight of products coming off a production line may need to legally be some value. By weighing some fraction of the products an average weight can be found, which will always be slightly different from the long term average. By using standard deviations, a minimum and maximum value can be calculated that the averaged weight will be within some very high percentage of the time (99.9% or more). If it falls outside the range then the production process may need to be corrected.

Example 8

Consider the average daily maximum temperatures for two cities, one inland and one on the coast. It is helpful to understand that the range of daily maximum temperatures for cities near the coast is smaller than for cities inland. Thus, while these two cities may each have the same average maximum temperature, the standard deviation of the daily maximum temperature for the coastal city will be less than that of the inland city as on any particular day, the actual maximum temperature is more likely to be farther from the average maximum temperature for the inland city than for the coastal one.

In finance, standard deviation is often used as a measure of the risk associated with price-fluctuations of a given asset (stocks, bonds, property, etc.), or the risk of a portfolio of assets, Standard deviation provides a quantified estimate of the uncertainty of future returns.

Unit summary

1. Variance measures how far a set of numbers is spread out. The variance is denoted and defined by

$$\sigma^2 = \frac{\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2}{n}$$

2. The standard deviation has the same dimension as the data, and hence is comparable to deviations from the mean. We define the standard deviation to be the square root of the variance. Thus, the standard deviation is denoted and defined by

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2}{n}} \text{ or } \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\overline{x}\right)^2}$$

3. The coefficient of variation measures variability in relation to the mean (or average) and is used to compare the relative dispersion in one type of data with the relative dispersion in another type of data. The coefficient of variation is

$$Cv = \frac{\sigma}{\overline{x}} \times 100$$

4. Application

A large standard deviation indicates that the data points can spread far from the mean and a small standard deviation indicates that they are clustered closely around the mean. Standard deviation is often used to compare real-world data against a model to test the model. Standard deviation is often used as a measure of the risk associated with price-fluctuations of a given asset (stocks, bonds, property, etc.), or the risk of a portfolio of assets. Standard deviation provides a quantified estimate of the uncertainty of future returns.

Revision exercise

1. The mean of 200 items was 50. Later on it was discovered that two items were misread as 92 and 8 instead of 192 and 88. Find the correct mean.
2. Calculate the mean and standard deviation of the following series:

x	1-10	11-20	21-30	31-40	41-50	51-60
Frequency	3	16	26	31	16	8

3. Find the mean of:
   a) 6, 10, 4, 13, 11, 9, 1, 6, 12 b) 193, 195, 202, 190, 189, 195
4. Find the mean and standard deviation of 25.2, 22.8, 22.1, 25.3, 24.6, 25.0, 24.3 and 22.7.

5. The mean height of a group of 5 people is $\bar{h} = 155\ cm$ and the standard deviation of their heights is *5 cm*.
   a) Calculate $\sum h$ and $\sum h^2$ for this data.
   b) If an extra person of height $165\ cm$ is added to the group, calculate the new mean and standard deviation of the heights.
6. The mean of the numbers 12, 18, 21, *c*, 13 is 17. Find the value of *c*.
7. The mean of 4 numbers is 5, and the mean of 3 different numbers is 12. What is the mean of the 7 numbers together?
8. The mean of *n* numbers is 5. If the number 13 is now included with the *n* numbers, the new mean is 6. Find the value of *n*.
9. The mean and variance of $x_1, x_2, x_3, ..., x_n$ are $\bar{x}$ and σ^2 respectively. State the mean and variance of $2-3x_1, 2-3x_2, 2-3x_3, ..., 2-3x_n$.
10. The mean daily maximum temperature at a fixed location for the month of January was $22^\circ C$ and the standard deviation of these daily maxima was $2^\circ C$.
    For the following February which was not a leap year, the mean and standard deviation was $4^\circ C$. Calculate the mean and standard deviation of the daily maximum temperatures for the two months combined.
11. If the mean of the following frequency distribution is 3.66, find the value of *a*

x	1	2	3	4	5	6
Frequency	3	9	a	11	8	7

12. For a set of 9 numbers $\sum(x-\bar{x})^2 = 234$. Find the standard deviation of the numbers.
13. For a set of 9 numbers $\sum(x-\bar{x})^2 = 60$ and $\sum x^2 = 285$. Find the mean of the numbers.
14. The mean of the numbers 3, 6, 7, *a*, 14 is 8. Find the standard deviation of the set of numbers.

15. Twenty values of a random variable have a mean of 15 and a variance of 1.5. Another thirty values of the same random variable have a mean of 14 and a variance of 1.4. Find the mean and variance of the combined fifty values.
16. Twenty values of a random variable have a mean value of 12.5 and a variance of 1.35. If two more values are added to the original 20 the mean remains at 12.5 but the variance is increased by 0.082. Find the values added.
17. Using the mean and standard of numbers 3, 5, 6, 8 and 10, find the mean and standard deviation of each of the following numbers:
    a) 6, 8, 9, 11, 13
    b) 9, 15, 18, 24, 30
    c) 2.7, 4.5, 5.4, 7.2, 9.0

    Find the mean and standard deviation of the numbers containing numbers which are 5% higher than those in original numbers.
18. The mean of 10 numbers is 8. If an eleventh number is now included in the results, the mean becomes 9. What is the value of the eleventh number?
19. The numbers a, b, 8, 5, 7 have a mean of 6 and a variance of 2. Find the value of a and b, if $a > b$.

Unit 10 Elementary probability

My goals

By the end of this unit, I will explain:

- Permutations and Arrangements.
- Combinations.
- Concepts of probability.
- Properties and formulas.
- Counting techniques.

Introduction

Probability is a common sense for scholars and people in modern days. It is the chance that something will happen-how likely it is that some event will happen. No engineer or scientist can conduct research and development works without knowing the probability theory. Some academic fields based on the probability theory are statistics, communication theory, computer performance evaluation, signal and image processing, game theory.... Some applications of the probability theory are character recognition, speech recognition, opinion survey, missile control, seismic analysis...

The theory of game of chance formed the foundations of probability theory, contained at the same time the principle for combinations of elements of a finite set, and thus establishes the traditional connection between combinatorial analysis and probability theory.

Required outcomes

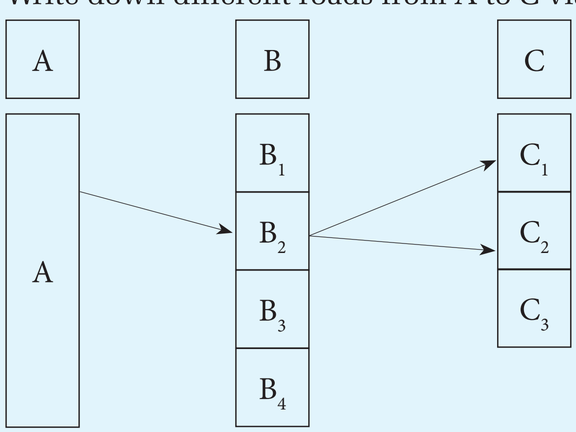
After completing this unit, the learners should be able to:

- solve problems involving factorial notation,
- determine the number of permutations of n different objects taken r at a time,
- determine the number of permutations of n different objects,
- determine the number of permutations of n different objects taken r at a time for a given condition,
- determine the number of combinations of r objects chosen from n different objects,
- determine the number of combinations of r objects chosen from n different objects for a given condition,
- use properties of combinations for finding coefficients in Pascal's triangle,
- solve problems involving permutations and combinations,
- find probability of events,
- determine probability of an equiprobable sample space

1. Permutations and arrangements

Activity 1

1. There are 4 roads joining A and B and 3 roads joining B and C. Write down different roads from A to C via B. How many are they?



c) Toss simultaneously a coin with two sides H and T and a die with six sides 1 through 6, write down different possible outcomes that can appear. How many are they?

Number of activity	Observation after tossing a coin	Observation after tossing a die	Outcome after tossing a coin and a die
1			
2			
3			
4			
‘ ‘			
Total number of different possible observations			

Combinatorial analysis is a **mathematical theory of counting**. Many problems in probability can be solved by simply counting the number of different ways in which a certain event can occur.

Principle of counting

Some terms:

Experiment: any human activity, like tossing a die.

Trial: small experiment contained in a large experiment. For example, tossing simultaneously a coin and a die may be regarded as an experiment consisting of two smaller experiments, tossing a coin (experiment 1) followed by tossing a die (experiment 2). Here experiment 1 and experiment 2 are called **trials**.

Outcome called **event**: a result of an experiment, like 4 point on a side of a die.

Example 1

For example, tossing simultaneously a coin and a die may be regarded as an experiment consisting of two smaller experiments, tossing a coin (experiment 1) followed by tossing a die (experiment 2). Here experiment 1 and experiment 2 are trials. Point on a side of a die is an outcome.

Notice:

Die (plural dice): small cube of plastic, ivory, bone, or wood, marked on each side with one to six sports, usually used in pairs in games of chance.

A die

Successive experiments

Basic product principle of counting

"If experiment 1 has m possible outcomes and if for each outcome of experiment 1, experiment 2 has n possible outcomes, then an experiment of performing experiment 1 and experiment 2 simultaneously, called **experiment 1 and 2**, has $m \times n$ possible outcomes."

Example 2

In tossing simultaneously a coin with two sides a and b and a die with six sides 1 through 6, how many possible outcomes will appear?

Solution

The tossing may be regarded as an experiment (experiment 1 & 2) consisting of two smaller experiments, tossing a coin (experiment 1) followed by tossing a die (experiment 2).
Experiment 1 has 2 outcomes.
Experiment 2 has 6 outcomes.
So experiment 1 & 2 has $2 \times 6 = 12$ outcomes.

Example 3

There are 5 roads joining A and B and 3 roads joining B and C how many different roads from A to C via B?

Solution

Here;
From A to B we have 5 roads.
From B to C we have 3 roads.
Thus, the number of roads from A to C via B is $3 \times 5 = 15$

Generalized product principle of counting

"If experiments 1 through k have n_1 through n_k outcomes, respectively, then the experiment 1, 2, 3 … and k has $n_1 \times n_2 \times \ldots \times n_k$ outcomes."

Note that this multiplication rule only applies when the experiments are independent, i.E, the choice made for one experiment does not affect the choice made for any of other experiments.

Example 4

A man has three choices of the way in which he travels to work; he can walk, go by car or go by train. How many different ways can he arrange his travel for the five working days in a week?

Solution

This man has 3 choices on Monday: walk, car or train
3 Choices on Tuesday: walk, car or train
Similarly on Wednesday, Thursday and Friday he has 3 choices.
Then there are 5 successive operations, each of which can be performed in 3 ways. Thus, the total number of choices he has is $3 \times 3 \times 3 \times 3 \times 3 = 243$ choices.

Example 5

A car license plate is to contain three letters of the alphabet, the first of which must be r, s, t or u, followed by three decimal digits. How many different license plates are possible?

Solution

The first letter can be chosen in 4 different ways, the second and third letters in 26 different ways each, and each of the three digits can be chosen in ten ways.

Hence there are $4\times 26\times 26\times 10\times 10\times 10=$ *2,704,000 plates possible.*

Exercise 1

1. There are 4 roads joining A and B and 5 roads joining B and C. How many different roads from A to C via B?
2. A car license plate is to contain two letters of the alphabet, the first of which must be A or B, followed by 6 decimal digits. How many different license plates are possible?

Permutations

Activity 2

Consider three letters **A**, **B** and **C** written in a row, one after another.

- Form all possible different words from three letters **A**, **B** and **C** (not necessarily sensible).

  In fact, each arrangement is a possible permutation of the letters A, B and C; for example ABC, ACB,…

- How many arrangements are they possible for three letters A, B and C?

How to calculate the number of permutations without having to list them all

From different arrangement of three letters **a**, **b** and **c**, the first letter to be written down can be chosen in 3 ways. The second letter can then be chosen in 2 ways because there are 2 remaining letters to be written down and the third letter can be chosen in 1 way because it is only one letter remaining to be written down. Thus, the three operations can be performed in $3\times 2\times 1=6$ ways.

This suggests the following fact:

Fact: Permutations of *n* objects

The number of different permutations of n different objects (unlike objects) in a row is

$n\times(n-1)\times(n-2)\times(n-3)\times\ldots\times2\times1$

A useful short hand of writing this operation is $n!$ (read ***n* factorial**). Then, $n!=n\times(n-1)\times(n-2)\times(n-3)\times\ldots\times2\times1$

Thus, $1!=1$, $2!=2\times1=2$, $3!=3\times2\times1=6$, $4!=4\times3\times2\times1=24$

$5!=5\times4\times3\times2\times1=120$ and so on.

Note that $0!=1$

Example 6

a) $\frac{6!}{2!4!}=\frac{6\times5\times4\times3\times2\times1}{2\times1\times4\times3\times2\times1}=15$ b) $\frac{7!}{4!2!}=\frac{7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1\times2\times1}=105$

Example 7

Five children are to be seated on a bench. In how many ways can they be seated? How many arrangements are they, if the youngest child has to sit at the left end of the bench?

Solution

Since there are five children, the first child can be chosen in 5 ways, the next child in 4 ways, the next in 3 ways, the next in 2 ways and the last in 1 way. Then, the number of ways is

$5!=5\times4\times3\times2\times1=120$.

Now, if the youngest child has to sit at the left end of the bench, this place can be filled in only 1 way. The next child can then be chosen in 4 ways, the next in 3 ways and so on. Thus, the number of total arrangement is $1\times4!=1\times4\times3\times2\times1=24$.

Example 8

Three different mathematics books and five other books are to be arranged on a bookshelf. Find:

a) the number of possible arrangements of the book.

b) the number of possible arrangements if the three mathematics books must be kept together.

Solution

We have 8 books altogether.

a) since we have 8 books altogether, the first book can be chosen in 8 ways, the next in 7 ways, the next in 6 ways and so on. Thus, the total arrangement is $8!=40{,}320$.

b) since the 3 mathematics books have to be together, consider these bound together as one book. There are now 6 books to be arranged and these can be performed in $6! = 720$.

Note that we have taken the three mathematics books as one book; these three books can be arranged in $3! = 6$ ways.

Thus, the total number of arrangements is $720 \times 6 = 4{,}320$.

Exercise 2

1. Four different English books, five different Biology books and ten other books are to be arranged on a bookshelf. Find
   a) the number of possible arrangements of the books.
   b) the number of possible arrangements if the three Biology books must be kept together?
2. Simplify

   a) $\frac{5!}{2!}$ b) $\frac{10!}{6!7!}$

Permutations of indistinguishable objects

Activity 3

Consider the arrangements of four letters in the word "**BOOM**".

- Write down all possible different arrangements.
- How many arrangements are they possible of four letters in the word "**BOOM**"?

In the same way,

- Write down all possible different arrangements of five letters in the word "CLASS".
- How many arrangements are they possible of five letters in the word "CLASS"?

Consider the arrangements of six letters in the word "**avatar**" (a title used for the movie).

We see that there are three **A**'s (or 3 alike letters).

- Let the three **A**'s in the word be distinguished as A_1, A_2 and A_3 respectively. Then all the six letters are different, so the number of permutations of them (called labeled permutations) is $n! = 6!$.

- However, consider each of the real permutations without distinguishing the three **A**'s, for example **W=RATAVA**.
- The following are all of the 6 (=3!) Labeled permutations among the 6! Ones, which come from permuting the three labeled **a**'s in **W=RATAVA**:

$\mathbf{RA_1TA_2VA_3, RA_1TA_3VA_2, RA_2TA_1VA_3, RA_2TA_3VA_1, RA_3TA_1VA_2, RA_3TA_2VA_1}$

- All these six labeled permutations should be considered as an identical real permutation, which is **W=RATAVA**.

Since each real permutation has six of such labeled permutations coming from the three **A**'s, we conclude that the desired number of real permutations is just $\frac{6!}{3!} = \frac{6\times5\times4\times3!}{3!} = 6\times5\times4 = 120$

This suggests the following fact:

Fact: permutations of indistinguishable objects

The number of different permutations of n objects with n_1 alike, n_2 alike, …, is $\frac{n!}{n_1!n_2!\ldots}$.

Note that **alike** means that the objects in a group are indistinguishable from one another.

Example 9

How many distinguishable six digit numbers can be formed from the digits 5, 4, 8, 4, 5, 4?

Solution

There are 6 letters in total with three 5's and two 4's. Then the required numbers are $h(x) = \begin{cases} 1, & x > 1 \\ 3, & x \le 1 \end{cases}$

Example 10

How many arrangements can be made from the letters of the word **TERRITORY**?

Solution

There are 9 letters in total with three **R**'s and two **T**'s.

Thus, we have $\frac{9!}{3!2!} = \frac{362880}{12} = 30{,}240$ arrangements.

Example 11

In how many different ways can 4 identical red balls, 3 identical green balls and a yellow ball be arranged in a row?

Solution

There are 8 balls in total with 4 red, 3 green and one yellow.

Thus, we have $\frac{8!}{4!\times 3!} = \frac{8\times 7\times 6\times 5}{3\times 2\times 1} = 280\ ways$

Exercise 3

1. How many arrangements can be made from the letters of the word **ENGLISH**?
2. How many arrangements can be made from the letters of English alphabet?
3. In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row, if discs of the same color are indistinguishable?

Circular arrangements

Activity 4

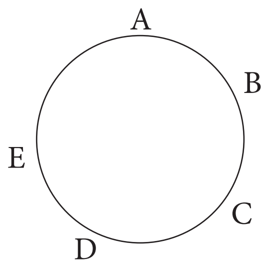
Take 5 different note books

- Put them on a circular table
- Fix one note book
- Try to interchange other 4 note books as possible
- How many different way obtained?

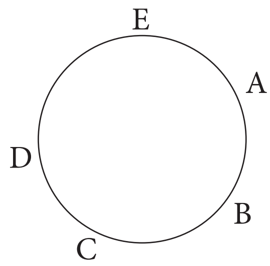
Remember that there is one note book will not change it place.

We have seen that if we wish to arrange n different things in a row, we have $n!$ possible arrangements. Suppose that we wish to arrange n things around a circular table. The number of possible arrangements will no longer be $n!$ because there is now no distinction between certain arrangements that were distinct when written in a row.

For example abcde is different arrangement from eabcd, but



is not a different arrangement from



With circular arrangement of this type, it is the relative positions of the items being arranged which is important. One item can therefore be fixed and the remaining items arranged around it.

The number of arrangements of n unlike things in a circle will therefore be $(n-1)!$. In those cases where clockwise and anticlockwise arrangements are not considered to be different, this reduces to $\frac{1}{2}(n-1)!$.

Example 12

Four men Peter, Rogers, Smith and Thomas are to be seated at a circular table. In how many ways can this be done?

Solution

Suppose peter is seated at some particular place. The seats on his left can be filled in 3 ways, the next seat can then be filled in 2 ways and the remaining seat in 1 way.

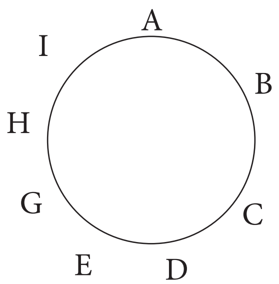
Thus, total number of arrangements is $f(0.99)$.

Example 13

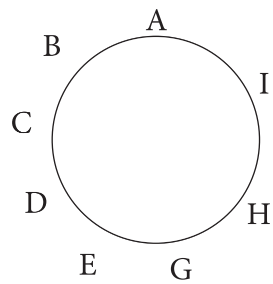
Nine beads, all of different colors are to be arranged on a circular wire. Two arrangements are not considered to be different if they appear the same when the ring is turned over. How many different possible arrangements are there?

Solution

When the ring is turned over, the arrangements



would appear as



When viewed from one side, these arrangements are only different in that one is a clockwise arrangement and the other is anticlockwise. If one bead is fixed, there are $(9-1)!=8!$ ways of arranging the remaining beads relative to the fixed one.

But, half of these arrangements will appear the same as the other half when the ring is turned over, because for every clockwise arrangement there is a similar anticlockwise arrangement. Hence, number of arrangements is $\frac{1}{2}8!=20160$

Exercise 4

1. Five men Eric, Peter, John, Smith and Thomas are to be seated at a circular table. In how many ways can this be done?
2. Eleven different books are placed on a circular table. In how many ways can this be done?

Mutually exclusive situation

Activity 5

When you go to a restaurant you will be asked if you wish to have a soup or juice. Will you pick one, the other or both?

Two experiments 1 and 2 are mutually exclusive, if when experiment 1 occurs, experiment 2 cannot occur. Likewise, if experiment 2 occurs, experiment 1 cannot occur.

Basic sum principle of counting

In such cases, the number of permutations of either experiment 1 or experiment 2 occuring can be obtained by adding the number of permutations of experiment 1 to the number of permutations of experiment 2.

This suggests the following result:

"If experiment 1 has m possible outcomes and if experiment 2 has n possible outcomes, then an experiment which might be experiment 1 or experiment 2, called **experiment 1 or 2**, has $m+n$ possible outcomes."

Example 14

In tossing an object which might be a coin (with two sides h and t) or a die (with six sides 1 through 6), how many possible outcomes will appear?

Solution

- The experiment may be t
- Tossing a coin (experiment 1) or tossing a die (experiment 2), or just experiment 1 or 2.
- So the number of outcomes is $2+6=8$ according to the above basic sum principle of counting.

Example 15

How many different four digit numbers, end in a 3 or a 4, can be formed from the figures 3,4,5,6 if each figure is used only once in each number.

Solution

We need the numbers that end in 3: the last digit can be chosen in one way, as it must be a 3, the first digit can then be chosen in 3 ways, the second in 2 ways and the third in 1 way. Thus, there are $1\times3\times2\times1=6$ numbers that end in a 3.

Similarly, there are $1\times3\times2\times1=6$ numbers that end in a 4.

The number that ends in a 3 cannot also end in a 4, so these are mutually exclusive situations.

Thus, there are $6+6=12$ numbers ending either in a 3 or in a 4.

Alternatively, this can be solved as follows:

The last digit can be chosen in 2 ways (3 or 4); the first digit can be chosen in 3 ways, the second in 2 ways and the third in 1 way, i.e, $2\times3\times2\times1=12$ numbers end either in a 3 or in a 4.

The number of permutations in which a certain experiment 1 occurs will clearly be mutually exclusive with those permutations in which that experiment does not occur. Thus,

$$\text{Number of permutations in which experiment 1 does not occur}$$
$$= \text{total number of permutation - number of permutations in which experiment 1 occurs}$$

Example 16

In how many ways can five people Smith, James, Clark, Brown and John, be arranged around a circular table in each of following cases:

a) Smith must sit next to Brown.

b) Smith must not sit next to Brown.

Solution

There are five people.

a) Since Smith and Brown must sit next to each other, consider these two bound together as one person. There are now, 4 people to seat. Fix one of these, and then the remaining 3 people can be seated in $3\times2\times1=6$ ways relative to the one who was fixed.

In each of these arrangements, smith and brown are seated together in a particular way. Smith and Brown could now change the seats giving another 6 ways of arranging the five people. Total number of arrangements is $6\times2=12$.

b) if Smith must not sit next to Brown, then this situation is a mutually exclusive with the situation in a).

Total number of arrangements of 5 people at a circular table is $(5-1)!=4!=24$.

Thus, if Smith must not sit next to Brown, the number of arrangements is $24-12=12$.

Generalised sum principle of counting

"If experiments 1 through k have respectively n_1 through n_k outcomes, then the experiment 1 or 2 or … or k has $n_1 + n_2 + ... + n_k$ outcomes.

Example 17

How many even numbers containing one or more digits can be formed from the digits 2, 3, 4, 5, 6 if no digit may be repeated?

Solution

Since the required numbers are even, last digit must 2 or 4 or 6. Note that there are 5 digits.

So we can form one digit, two digits, three digits, four digits or five digits as follows:

One digit: 2 or 4 or 6. That is 3 numbers.

Two digits: 3 ways to choose the last and 4 ways to choose the first. That is $3 \times 4 = 12$ numbers.

Three digits: 3 ways to choose the last, 4 ways to choose the first and 3 ways to choose the second. That is $3 \times 4 \times 3 = 36$

Four digits: 3 ways to choose the last, 4 ways to choose the first, 3 ways to choose the second and 2 ways to choose the fourth. That is $3 \times 4 \times 3 \times 2 = 72$

Five digits: 3 ways to choose the last, 4 ways to choose the first, 3 ways to choose the second, 2 ways to choose the fourth and 1 way to choose the fifth. That is $3 \times 4 \times 3 \times 2 \times 1 = 72$

Adding, we have $3 + 12 + 36 + 72 + 72 = 195$ even numbers in total.

Exercise 5

1. How many odd numbers containing one or more digits can be formed from the digits 2, 3, 4, 5, 6, 7 if no digit may be repeated?
2. How many even numbers containing 2 digits can be formed from the digits 2, 3, 4 if no digit may be repeated?
3. How many numbers containing three digits can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8 if no digit may be repeated?

Distinguishable permutations

Activity 6

Make a selection of any three letters from the word "**KNOW**" and fill them in 3 empty spaces

Use a box like this for empty spaces		

Write down all different possible permutations of 3 letters selected from the letters of the word "**KNOW**".

How many are they?

Consider the number of ways of placing 3 of the letters a, b, c, d, e, f, g in 3 empty spaces.

The first space can be filled in 7 ways, the second in 6 ways and the third in 5 ways. Therefore, there are $7\times6\times5$ ways of arranging 3 letters taken from 7 letters. This is the number of permutations of 3 objects taken from 7 and it is written ${}^7P_3$. So ${}^7P_3 = 7\times6\times5 = 210$.

Note that the order in which the letters are arranged is important: abc is a different permutation from acb.

Now, $7\times6\times5$ could be written $\frac{7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1}$

I.E ${}^7P_3 = \frac{7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1} = \frac{7!}{4!} = \frac{7!}{(7-3)!}$

This suggests the following fact:

Fact: Permutations of *r* objects selected from *n* ones

The number of different permutations of r unlike objects selected from n different objects is ${}^nP_r = \frac{n!}{(n-r)!}$ or we can use the denotation $P_r^n = \frac{n!}{(n-r)!}$ or $P(n,r) = \frac{n!}{(n-r)!}$

Note that if $r = n$, we have ${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$ which is the ways of arranging n unlike objects.

Example 18

How many permutations are there of 3 letters chosen from eight letters of the word relation ?

Solution

We see that all those eight letters are distinguishable (unlike). So the required arrangements are given by

$${}^8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 336.$$

Example 19

How many permutations are there of 2 letters chosen from letters a, b, c, d, e?

Solution

There are 5 letters which are distinguishable (unlike). So the required arrangements are given by ${}^5P_2 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = 20$.

Example 20

How many different arrangements are there of 3 letters chosen from the word COMBINATION?

Solution

There are 11 letters including two O's, two I's and two N's. to find the total number of different arrangements we consider the possible arrangements as four mutually exclusive situations.

a) Arrangements in which all 3 letters are different: there are ${}^{8}P_{3} = 336$

b) Arrangements containing two O's and one other letter: the other letter can be one of seven letters (C, M, B, I, N, A or T) and can appear in any of the three positions (before the two O'S, between the two O's, or after the two O's). i.e $3 \times 7 = 21$ arrangements containing two O's and one other letter.

c) Arrangements containing two I's and one other letter: by the same reasoning in b) there will be $3 \times 7 = 21$ arrangements containing two I's and one other letter.

d) Arrangements containing two N's and one other letter: by the same reasoning in b) there will be $3 \times 7 = 21$ arrangements containing two N's and one other letter.

Thus the total number of arrangements of 3 letters chosen from the word COMBINATION will be $336 + 21 + 21 + 21 = 399$

Exercise 6

1. How many permutations are there of 4 letters chosen from letters of the word ENGLISH?
2. How many permutations are there of 2 letters chosen from letters of the word PACIFIC?
3. How many permutations are there of 5 letters chosen from letters A, B, C, D, E, F, and G.
4. How many permutations are there of 10 letters chosen from English alphabet.

2. Combinations

Activity 7

Take 3 different mathematics books and 4 unlike English books so that you will have 7 books altogether. Form different groups each containing 2 mathematics books and 2 English books. How many groups obtained?

From permutation of r unlike objects selected from n different objects, we have seen that the order in which those objects are placed is important. But when considering the number of combinations of r unlike objects selected from n different objects, the order in which they are placed is not important. For example, the one combination **ABC** gives rise to 3! Permutations: **ABC, ACB, BCA, BAC, CAB, CBA**.

Consider the number of permutations of 3 letters selected from the 7 letters **A, B, C, D, E, F, G**.

That is;

$${}^{7}P_{3} = \frac{7!}{(7-3)!} = \frac{7!}{4!}.$$

If we need the combinations of 3 letters selected from those 7 letters, we will take this number of permutations divided by 3! because each permutation gives rise to 3! Permutations.

That is, the number of combinations of 3 letters selected from those 7 letters is $\frac{{}^{7}P_{3}}{3!} = \frac{\frac{7!}{4!}}{3!} = \frac{7!}{4!3!} = \frac{7!}{(7-3)!3!}$.

This number is denoted by ${}^{7}C_{3}$.

Thus, the number of combinations of 3 letters selected from those 7 unlike letters is ${}^{7}C_{3} = \frac{7!}{(7-3)!3!} = \frac{7!}{4!3!} = 35$.

This suggests the following fact:

Fact: Basic formula for combinations

The number of different groups of r items that could be formed from a set of n distinct objects with the order of selections being ignored is

$$^nC_r = \frac{n!}{(n-r)!r!}.$$

We can write $^nC_r = \frac{^nP_r}{r!}$

^nC_r Is sometimes denoted by C_r^n or ^nc_r or $\binom{n}{r}$ or $C(n,r)$.

Note that the objects selected to be in a group are regarded as indistinguishable (unlike).

Example 21

From a group of 5 men and 7 women, how many different committees consisting of 2 men and 3 women can be formed?

Solution

- Experiment 1: select 2 men from 5.
- Experiment 2: select 3 women from 7.
- Experiment of forming a committee: experiment 1 & 2.
- Number of possible outcomes of experiment 1 is $^5C_2 = \frac{5!}{2!3!} = \frac{5\times4\times3!}{2\times3!} = 10$.
- Number of possible outcomes of experiment 2 is $^7C_3 = \frac{7!}{3!4!} = \frac{7\times6\times5\times4!}{6\times4!} = 35$.
- Number of possible outcomes of experiment 1 and 2 is

  $^5C_2 \times {^7C_3} = 10\times35 = 350$ by the basic product principle of counting.
- That is, the desired number of possible outcomes of the experiment of forming a committee is 350.

Example 22

A committee of three men and one woman is obtained from five men and three women. In how many ways can the members be chosen?

Solution

Three men can be selected from five men, i.e

$$^5C_3 = \frac{5!}{(5-3)!3!} = \frac{5!}{2!3!} \text{ ways}$$

One woman can be selected from three women, i.e

$$^3C_1 = \frac{3!}{(3-1)!1!} = \frac{3!}{2!1!} \text{ ways}$$

By the basic product principle of counting, there are

$^5C_3 \times {}^3C_1 = \frac{5!}{2!3!} \times \frac{3!}{2!1!} = \frac{5!}{2!2!} = 30$ ways of selecting the Committee.

Fact: Two identities about computations of combinations

The following two identities are true:

- $^nC_r = {}^nC_{n-r}$

In fact,

$$^nC_r = \frac{n!}{(n-r)!r!}$$

$$= \frac{n!}{(n-r)!(n-n+r)!}$$

$$= \frac{n!}{(n-r)![n-(n-r)]!}$$

$$= {}^nC_{n-r}$$

- Pascal's identity: $^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$

In fact,

$$^nC_r + {}^nC_{r-1} = \frac{n!}{(n-r)!r!} + \frac{n!}{(n-[r-1])!(r-1)!}$$

$$= \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!}$$

$$= \frac{n!(r-1)!(n-r+1)! + n!(n-r)!r!}{(n-r)!r!(r-1)!(n-r+1)!}$$

$$=\frac{n!(r-1)!(n-r+1)!+n!(n-r)!r!}{(n-r)!r!(r-1)!(n-r+1)!}$$

$$=\frac{n!}{r!(n-r+1)!}\left[\frac{(r-1)!(n-r+1)!+(n-r)!r!}{(n-r)!(r-1)!}\right]$$

$$=\frac{n!}{r!(n-r+1)!}\left[\frac{(r-1)!(n-r+1)!+(n-r)!r(r-1)!}{(n-r)!(r-1)!}\right]$$

$$=\frac{n!}{r!(n-r+1)!}\left[\frac{(r-1)!\left[(n-r+1)!+(n-r)!r\right]}{(n-r)!(r-1)!}\right]$$

$$=\frac{n!}{r!(n-r+1)!}\left[\frac{(n-r+1)!+(n-r)!r}{(n-r)!}\right]$$

$$=\frac{n!}{r!(n-r+1)!}\left[\frac{(n-r+1)(n-r)!+(n-r)!r}{(n-r)!}\right]$$

$$=\frac{n!}{r!(n-r+1)!}\left[\frac{(n-r)!\left[(n-r+1)+r\right]}{(n-r)!}\right]$$

$$=\frac{n!(n+1)}{r!(n-r+1)!}$$

$$=\frac{(n+1)!}{r!(n+1-r)!}$$

$$={}^{n+1}C_r$$

Exercise 7

1. A committee of four men and two women is obtained from 10 men and 12 women. In how many ways can the members be chosen?
2. A group containing 4 Mathematics books and 5 Physics books is formed from 9 Mathematics books and 10 Physics books. How many groups can be formed?

Summary- arrangements, permutations and combinations:

The number of ways of arranging n unlike objects in a row.	$n!$
The number of ways of arranging in a row n objects of/ with n_1 alike, n_2 alike, ..., n_r alike.	$\frac{n!}{n_1!n_2!...}$
The number of ways of arranging n unlike objects in a ring when clockwise and anticlockwise arrangements are different.	$(n-1)!$
The number of ways of arranging n unlike objects in a ring when clockwise and anticlockwise arrangements are the same.	$\frac{(n-1)!}{2}$
The number of permutations of r objects taken from n unlike objects.	${}^nP_r = \frac{n!}{(n-r)!}$
The number of combinations of r objects taken from n unlike objects.	${}^nC_r = \frac{n!}{(n-r)!r!}$

Binomial expansion

Activity 8

Expand the expressions $(a+b)^2$

Since $(a+b)^3 = (a+b)^2(a+b)$ *and* $(a+b)^4 = (a+b)^3(a+b)$

Expand $(a+b)^3$ and $(a+b)^4$

Once more find the expansion of $(a+b)^5$.

Complete the following table

Power	Coefficient of powers of ***a*** and ***b***					Binomial expression
0						$(a+b)^0$
1						$(a+b)^1$
2						$(a+b)^2$
3						$(a+b)^3$
4						$(a+b)^4$

Pascal's triangle

Pascal's triangle is a triangular array of the binomial coefficients. The rows of pascal's triangle are conventionally enumerated starting with row $n=0$ at the top. The entries in each row are numbered from the left beginning with $r=0$ and are usually staggered relative to the numbers in the adjacent rows.

The elements of Pascal's triangle are the number of combinations of r objects chosen from n unlike objects. That is ^nC_r. This triangle is constructed by the **Pascal's identity:**

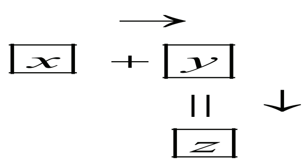
$$^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$$

Or

$$^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

Or

$$^{n+1}C_{r+1} = {}^nC_r + {}^nC_{r+1}$$



Here, $z = {}^{n+1}C_r$,

$y = {}^nC_r$ And

$x = {}^nC_{r-1}$

R \ n	0	1	2	3	4	5	…
0	$^0C_0=1$						
1	$^1C_0=1$	$^1C_1=1$					
2	$^2C_0=1$	$^2C_1=2$	$^2C_2=1$				
3	$^3C_0=1$	$^3C_1=3$	$^3C_2=3$	$^3C_3=1$			
4	$^4C_0=1$	$^4C_1=4$	$^4C_2=6$	$^4C_3=4$	$^4C_4=1$		
5	$^5C_0=1$	$^5C_1=5$	$^5C_2=10$	$^5C_3=10$	$^5C_4=5$	$^5C_5=1$	
⋮							

A simple construction of this triangle proceeds in the following manner:

- On row 0, write only the number 1.
- Then, to construct the elements of following rows, add the number above and to the left with the number above to the right to find the new value.
- If either the number to the right or left is not present, substitute a zero in its place. For example, the first number in the first row is $0+1=1$, whereas the numbers 1 and 3 in the third row are added to produce the number 4 in the fourth row.

An element of pascal's triangle, ${}^{n}C_{r}$, is the coefficients of any term in the expansion of $(a+b)^{n}$ where *r* is the exponent of either *a* or *b*.

Consider the product $(a+b)^{n}=(a+b)(a+b)(a+b)...(a+b)$. If this product is multiplied out, each term of the answer will be of the form $c_1c_2c_3...c_k...c_n$ where, for all k, c_k is either *a* or *b*.

Thus, if $c_k=a$, for all *k* we obtain the term a^{k}. If $c_k=b$ for one of the terms and $c_k=a$ for the rest, we obtain terms such as $b\times a\times a\times...\times a\times a$, $a\times b\times a\times...\times a\times a$, ..., $a\times a\times a\times...\times b\times a$ $b\times a\times a\times...\times a\times b$, and their sum is $na^{n-1}b$.

If $c_k=b$ for *r* of the terms and $c_k=a$ for the rest, we obtain a number of terms of the form $a^{n-r}b^{r}$.

The number of such terms is the number of ways in which *r* of the form $c_1c_2c_3...c_n$ can be selected as equal to *b*) this number is $\begin{pmatrix} n \\ r \end{pmatrix}$, which is $\begin{pmatrix} n \\ r \end{pmatrix}={}^{n}C_{r}=\frac{n!}{(n-r)!r!}$.

Thus, ${}^{n}C_{r}=\frac{n!}{(n-r)!r!}$ is the coefficient of $a^{n-r}b^{r}$ in the

Expansion of $(a+b)^n$.

This suggests the following theorem.

Binomial theorem

For every integer $n \geq 1$, $(a+b)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r} b^r$

The following properties of the expansion of $(a+b)^n$ should be observed:

a) There are $n+1$ terms.

b) The sum of the exponents of a and b in each term is n.

c) The exponents of a decrease term by term from n and 0; the exponent of b increase term by term from 0 to n.

d) The coefficient of any term is ${}^nC_r$ where r is the exponent of either a or b.

e) The coefficients of terms equidistant from the end are equal.

Example 23

$$(a+b)^2 = \sum_{r=0}^{2} {}^2C_r a^{2-r} b^r = {}^2C_0 a^2 b^0 + {}^2C_1 a^{2-1} b^1 + {}^2C_2 a^{2-2} b^2 = a^2 + 2ab + b^2$$

Example 24

$$(a-b)^3 = (a+(-b))^3 = \sum_{r=0}^{3} {}^3C_r a^{3-r} (-b)^r = {}^3C_0 a^3 (-b)^0 + {}^3C_1 a^{3-1} (-b)^1 + {}^3C_2 a^{3-2} (-b)^2 + {}^3C_3 a^{3-3} (-b)^3$$

$$= a^3 - 3a^2b + 3ab^2 - b^3$$

Example 25

Find the coefficient of x^3 in the expansion of $(2x-1)^5$

Solution

The term in x^r is ${}^5C_r(2x)^{5-r}(-1)^r = {}^5C_r 2^{5-r}x^{5-r}(-1)^r$ and so the term in x^3 has $r=2$.

The coefficient of this term is ${}^5C_3 2^3(-1)^2 = 80$.

Example 26

Find the coefficient of x^3 in the expansion of $\left(x^2 - \frac{1}{x}\right)^6$

Solution

The term in x^r is will be given by ${}^6C_r\left(x^2\right)^{6-r}\left(-\frac{1}{x}\right)^r$ which can be written as ${}^6C_r x^{12-2r}\frac{(-1)^r}{x^r} = {}^6C_r x^{12-3r}(-1)^r$ and so the term in x^3 has $r=3$.

The coefficient of this term is ${}^6C_3(-1)^3 = -20$.

Exercise 8

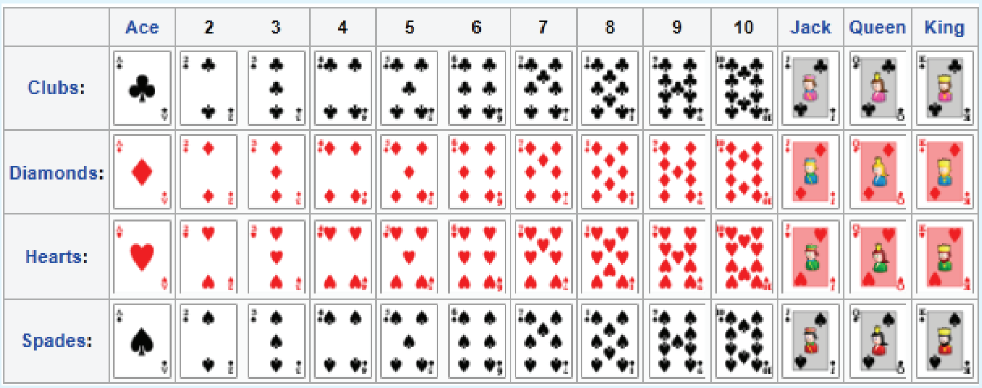
Find the coefficient of

1. x^2 in the expansion of $(4x+1)^6$
2. x^3 in the expansion of $\left(x+\frac{1}{x}\right)^4$
3. x^6 in the expansion of $(9x-3)^{10}$
4. Expand $(x+4)^7$
5. Expand $(2x-3)^3$

3. Concepts of probability

Activity 9

Consider the deck of 52 playing cards.



1. Suppose that you are choosing one card;

   (i) How many cards can be chosen?

   (ii) How many kings can be chosen?

   (iii) How many aces of hearts can be chosen?

2. If a spade card has been declared to be a trump card. How many trumps of Jack can be chosen?

Probability is the chance that something will happen.

The concept of probability can be illustrated in the context of game of 52 playing cards. In a park of deck of 52 playing cards, cards are divided into four suits of 13 cards each. If a player selects a card at random (by simple random sampling), then each card has the same probability of being selected.

In most sampling situations we are generally not concerned with sampling a specific individual but instead we concern ourselves with the probability of sampling certain types of individuals. For example, for our case of a deck of 52 playing cards, what is the probability of selecting a queen?

Selecting a queen is the **event** and selecting any card from 52 cards is an **experiment.**

Random experiments and events

A **random experiment** is an experiment that, atleast theoretically, may be repeated as often as we want and whose outcome cannot be predicted, the roll of a die. Each time experiment is repeated, an **elementary outcome** is obtained. The set of all elementary outcomes of a random experiment is called the **sample space**, which is denoted by Ω. Sample space may be discrete or continuous.

Discrete sample space:

- Firstly, the number of possible outcomes is **finite**.
- Secondly, if the number of possible outcomes is **countably infinite**, which means that there is an infinite number of possible outcomes, but the outcomes can be put in a one-to-one correspondence with the positive integers.

Example 27

If a die is rolled and the number that shows up is noted, then $\Omega = \{1, 2, 3, ..., 6\}$.

A die

Example 28

If a die is rolled until a "6" is obtained, and the number of rolls made before getting first "6" is counted, then we have that $\Omega = \{0, 1, 2, 3, ...\}$.

Continuous sample space:

If the sample space contains one or more intervals, the sample space is then **uncountable infinite**.

Example 29

A die is rolled until a "6" is obtained and the time needed to get this first "6" is recorded. In this case, we have that

$\Omega = \{t \in \mathbb{R} : t > 0\} = (0, \infty)$.

An **event** is a set of elementary outcomes. That is, it is a subset of the sample space.

In particular, every elementary outcome is an event, and so is the sample space itself.

Remarks:

- An elementary outcome is sometimes called a **simple event**, whereas a **compound** event is made up of at least two elementary outcomes.
- To be precise, we should distinguish between the elementary outcome w, which is an element of Ω and the elementary event $\{w\} \subset \Omega$.
- The events are denoted by A, B, C and so on.

Example 30

Consider the experiment that consists in rolling a die and recording the number that shows up. Let A be the event "the even number is shown" and B be the event "the odd number less than 5 is shown". Define the events A and B.

Solution

We have the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$.

$A = \{2, 4, 6\}$ And $B = \{1, 3\}$

Definitions

- Two or more events which have an equal probability of occurrence are said to be **equally likely**, i.e. If on taking into account all the conditions, there should be no reason to accept any one of the events in preference over the others. Equally, likely events may be simple or compound events.

- Two events, A and B are said to be **incompatible** (or **mutually exclusive**) if their intersection is empty. We then write that $A \cap B = \varnothing$.
- Two events, A and B are said to be **exhaustive** if they satisfy the condition $A \cup B = \Omega$.
- An event which is sure to occur at every performance of an experiment is called a **certain event** connected with the experiment. For example, "Head or Tail' is a certain event connected with tossing a coin. Face-1 or face-2, face-3, ……, face-6 is a certain event connected with throwing a die.
- An event which cannot occur at any performance of the experiment is called an **impossible event.** Following are such examples
  - i. 'Seven' in case of throwing a die.
  - ii. 'Sum=13' in case of throwing a pair of dice.
- Two events are said to be **equivalent or identical** if one of them implies and implied by other. That is, the occurrence of one event implies the occurrence of the other and vice versa. For example, "even face" and "face-2" or "face-4" or "face-6" are two identical events.
- The outcomes which make necessary the happening of an event in a trial are called **favourable events.** For example; if two dice are thrown, the number of favourable events of getting a sum 5 is four, i.e., (1, 4), (2, 3), (3, 2) and (4, 1).

Example 31

Consider the experiment that consists in rolling a die and recording the number that shows up.

We have that $\Omega = \{1,2,3,4,5,6\}$.

We define the events

$A=\{1,2,4\}$, $B=\{2,4,6\}$, $C=\{3,5\}$, $D=\{1,2,3,4\}$ And $E=\{3,4,5,6\}$

We have;

$A\cup B=\{1,2,4,6\}$, $A\cap B=\{2,4\}$,

$A\cap C=\varnothing$ and $D\cup E=\Omega$.

Therefore, A and C are incompatible events.

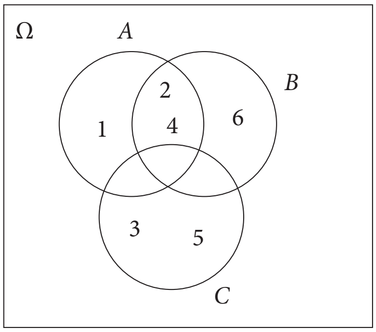
D and E are exhaustive events.

Moreover, we may write that $A'=\{3,5,6\}$, where the symbol A' denotes the **complement** of the event A.

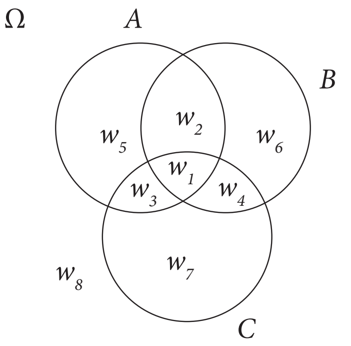
This suggests the following definition:

If E is an event, then E' is the event which occurs when E does not occur. Events E and E' are said to be **complementary events.**

To represent a sample space and some events, we often use a **venn diagram** as in figure below:



In general, for three events we have the following diagram



$w_1=A\cap B\cap C$

$w_2=A\cap B\cap C'$

$w_3=A\cap B'\cap C$

$w_4=A'\cap B\cap C$

$w_5=A\cap B'\cap C'$

$w_6=A'\cap B\cap C'$

$w_7=A'\cap B'\cap C$

$w_8=A'\cap B'\cap C'$

Example 32

In the experiment of tossing a coin:

Where;

(i) A : the event of getting a "head" and

(ii) B : the event of getting a "tail"

Events "A" and "B" are said to be equally likely events. Both the events have the same chance of occurrence.

Example 33

Consider the experiment that consists in choosing at random from the list 2, 3, 5, 7, 11, 13, 17, 19. The event that the chosen number is not prime number is impossible events since all numbers are prime numbers.

Example 34

Consider the experiment that consists in rolling a die. The event of rolling a number that is not 8 is a certain event.

Exercise 9

1. A box contains 5 red, 3 blue and 2 green pens. If a pen is chosen at random from the box, then which of the following is an impossible event?
   a) Choosing a red pen.
   b) Choosing a blue pen.
   c) Choosing a yellow pen.
   d) None of the above.
2. A spinner has 9 equal sectors numbered 1 to 9. If you spin the spinner, then which of the following is a certain event?
   a) Landing on a number less than 9.
   b) Landing on a number less than 12.
   c) Landing on a number greater than 1.
   d) None of the above.

3. Which of the following are mutually exclusive events when a day of the week is chosen at random?
   a) Choosing a Monday or choosing a Wednesday.
   b) Choosing a Saturday or choosing a Sunday.
   c) Choosing a weekday or choosing a weekend day.
   d) All of the above.
4. A die is tossed, tell whether the following events are exhaustive or not.
   a) X = Get prime number ; Y = Get multiple of 2 ; Z = Get 1.
   b) X = Get prime numbers; Y = Get composite numbers; Z = Get 1.
   c) X = an odd number; Y = an even number.

4. Properties and formulas

Activity 10

Consider the letters of the word "PROBABILITY".

a) How many letters are in this word

b) How many vowels are in this word? What is the ratio of numbers of vowels to the total number of letters?

c) How many consonants are in this word? what is the ratio of the numbers of consonants to total number of letters

d) Let A be the set of all vowels and B the set of all consonants. Find

(i) $A\cap B$ (ii) $A\cup B$ (iii) A' (iv) B'

The probability of an event $A\subset\Omega$, is a real number obtained by applying to A the function P defined by

$$P(A)=\frac{number\ of\ \ favorable\ outcomes}{number\ of\ \ possible\ outcomes}=\frac{\#A}{\#\Omega}$$

Theorem 1

Suppose that an experiment has only a finite number of equally likely outcomes. If e is an event, then $0\le P(A)\le 1$.

Note that if $A=\Omega$, then $P(A)=1$ and $P(\Omega)=1$ (the event is certain to occur), and if $A=\varnothing$ then $P(A)=0$ (the event cannot occur).

Example 35

A letter is chosen from the letters of the word "MATHEMATICS". What is the probability that the letter chosen is an "a"?

Solution

Since two of the eleven letters are *a*'s, we have two favorable outcomes.

There are eleven letters, so we have 11 possible outcomes.

Thus, the probability of choosing a letter a is $\frac{2}{11}$.

Theorem 2

$P(E)=1-P(E')$ Where E and E' are complementary events.

Consider two different events, A and B, which may occur when an experiment is performed.

- The event $A\cup B$ is the event which occurs if A or B or both A and B occur, i.e, at least one of A and B occurs.
- The event $A\cap B$ is the event which occurs if A and B occur.
- The event $A-B$ is the event which occurs when A occurs and B does not occur.
- The event A' is the event which occurs when A does not occur.

Note that if $A=A_1\cup A_2\cup A_3\cup\ldots\cup A_n$, where $A_1, A_2, A_3, \ldots, A_n$ are **incompatible events**, then we may write that

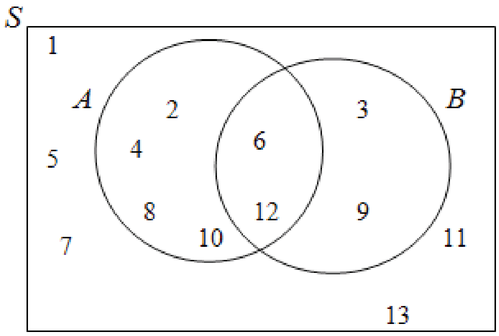
$P(A)=\sum_{i=1}^{n} P(A_i)$ for $n=2,3,\ldots$

Example 36

An integer is chosen at random from the set

$S=\{x: x\in\mathbb{Z}^+, x<14\}$. Let A be the event of choosing a multiple of 2 and let B be the event of choosing a multiple of 3. Find $P(A\cup B)$, $P(A\cap B)$ and $P(A-B)$.

Solution



From the diagram, $\#S=13$

$A\cup B=\{2,3,4,6,8,9,10,12\}\Rightarrow \#(A\cup B)=8,$ thus $P(A\cup B)=\frac{8}{13}$

$A\cap B=\{6,12\}\Rightarrow \#(A\cap B)=2,$ thus $P(A\cap B)=\frac{2}{13}$

$A-B=\{2,4,8,10\}\Rightarrow \#(A-B)=4,$ thus $P(A-B)=\frac{4}{13}$

Exercise 10

1. A letter is chosen from the letters of the word "**MATHEMATICS**". What is the probability that the letter chosen is

   a) M? b) T?

2. An integer is chosen at random from the set

   $S=\{\text{all positive integers less than } 20\}$. Let A be the event of choosing a multiple of 3 and let B be the event of choosing an odd number. Find;

   a) $P(A\cup B)$ b) $P(A\cap B)$ c) $P(A-B)$

Sum law

Activity 11

An integer is chosen at random from the set $S = \{\text{all positive integers less than } 10\}$. Let A be the event of choosing a multiple of 3 and let B be the event of choosing an odd number. Find

a) $P(A \cup B)$ b) $P(A) + P(B) - P(A \cap B)$

What can you say about result in a and result in b?

Theorem 3

If A and B are events from a sample space Ω, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

This is known as the **addition law** of probability.

Example 37

A card is drawn from a deck of 52 playing cards. If A is an event of drawing an ace and B is an event of drawing a spade. Find;

$$P(A), P(B), P(A \cap B), P(A \cup B)$$

Solution

There are 4 aces, then $P(A) = \frac{4}{52} = \frac{1}{13}$

There are 13 spades, then $P(B) = \frac{13}{52} = \frac{1}{4}$

There is 1 ace of spades, then $\#(A \cap B) = 1$ and $P(A \cap B) = \frac{1}{52}$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{1}{13} + \frac{1}{4} - \frac{1}{52} \\ &= \frac{16}{52} \\ &= \frac{4}{13} \end{aligned}$$

On the other hand, there are 4 aces and 13 spades but also 1 ace of spades. Then

$A \cup B = 16$ And $P(A \cup B) = \frac{16}{52} = \frac{4}{13}$

Example38

In a group of 20 adults, 4 out of the 7 women and 2 out of the 13 men wear glasses. What is the probability that a person chosen at random from the group is a woman or someone who wears glasses?

Solution

Let A be the event: "the person chosen is a woman".

B be the event: "the person chosen wears glasses".

Now,

There are 7 women, then $P(A) = \frac{7}{20}$

There are 6 persons who wear glasses, then $P(B) = \frac{6}{20}$

There are 4 women who wear glasses, then $P(A \cap B) = \frac{4}{20}$

The probability that a person chosen at random from the group is a woman or someone who wears glasses is given by

$P(A \text{ or } B)$ which is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{7}{20} + \frac{6}{20} - \frac{4}{20}$$

$$= \frac{9}{20}$$

On the other hand:

There are 7 women and 6 persons who wear glasses but also 4 women who wear glasses. Then

$A \cup B = 9$ And $P(A \cup B) = \frac{9}{20}$

Exercise 11

1. A fair die is rolled, what is the probability of getting a even number or prime number?
2. Two fair dice are rolled, what is the probability of getting a sum that is divisible by 2 or 4?

Mutually exclusive events

Activity 12

A book is drawn from a bookshelf containing 15 books of which 5 are Mathematics books and 10 are English books. If A is the event: "a book is a Mathematics book" and B is the event: "a book is an English book", find:

a) $P(A)$ b) $P(B)$ c) $P(A \cap B)$

d) $P(A)+P(B)$ e) $P(A \cup B)$

Compare your results from d to e

Events A and B are said to be **mutually exclusive** (or **incompatible**) events if they are disjoint, i.e, they cannot occur at the same time. In this case $A \cap B = \varnothing$ and the law of addition reduces to $P(A \cup B) = P(A) + P(B)$.

Example 39

A pen is drawn from a basket containing 10 pens of which 5 are red and 3 are black. If a is the event: "a pen is red" and b is the event: "a pen is black", find $P(A), P(B), P(A \cup B)$.

There are 5 red pens, then $P(A) = \frac{5}{10} = \frac{1}{2}$

There are 3 black pens, then $P(B) = \frac{3}{10}$

Since the pen cannot be red and black at the same time, then $A \cap B = \varnothing$ and two events are mutually exclusive so

$$P(A \cup B) = P(A) + P(B)$$
$$= \frac{1}{2} + \frac{3}{10}$$
$$= \frac{8}{10}$$
$$= \frac{4}{5}$$

Example 40

A card is drawn at random from an ordinary pack of 52 playing cards. Find the probability that the card drawn is a club or a diamond.

Solution

There are 13 clubs, then $P(\text{club}) = \frac{13}{52}$

There are 13 diamonds, then $P(\text{diamond}) = \frac{13}{52}$

Since a card cannot be both a club and a diamond, $P(\text{club} \cap \text{diamond}) = 0$

Therefore,

$$P(\text{a club or a diamond}) = P(\text{club}) + P(\text{diamond})$$
$$= \frac{13}{52} + \frac{13}{52}$$
$$= \frac{26}{52}$$
$$= \frac{1}{2}$$

Exercise 12

1. If A and B are mutually exclusive events, given the probability of A and B as $\frac{1}{5}$ and $\frac{1}{3}$ respectively, find the probability of at least any one event occurring at a time.
2. If X and Y are two events, the probability of the happening of X or Y is $\frac{7}{10}$ and the probability of X is $\frac{1}{3}$. If X and Y are mutually exclusive find the probability of Y?

Exhaustive events

Activity 13

An integer is chosen at random from the set $S = \{\text{all positive integers less than } 15\}$. Let A be the event of choosing an odd number and let B be the event of choosing an even number. Find:

a) $P(A \cup B)$ b) $P(S)$

What can you say about result in a. and result in b.?

If two events *a* and *b* are such that $A \cup B = \Omega$ then $P(A \cup B) = 1$ and then these two events are said to be exhaustive.

Generally, given a finite sample space, say $\Omega = \{A_1, A_2, A_3, ..., A_n\}$ we can find a finite probability by assigning to each point $A_i \in \Omega$ a real number p_i, called the probability of A_i, satisfying the following:

a) $p_i \geq 0$ for all integers i, $1 \leq i \leq n$ b) $\sum_{i=1}^{n} P_i = 1$.

If *E* is an event, then the probability $P(E)$ is defined to be the sum of the probabilities of the sample points in *E*.

Example 41

A coin is weighted so that heads is three times as likely to appear as tails. Find $P(H)$ and $P(T)$.

Solution

Let $P(T)=p_1$, then $P(H)=3p_1$.
But $P(H)+P(T)=1$
Therefore

$$3p_1+p_1=1 \Leftrightarrow 4p_1=1 \Rightarrow p_1=\frac{1}{4}$$

Thus, $P(H)=\frac{3}{4}$ and $P(T)=\frac{1}{4}$.

Example 42

A die is thrown once. Let A be the event: "the number obtained is less than 5" and B be the event: "the number obtained is greater than 3". Find probability of $A\cup B$.

Solution

Here $A=\{1,2,3,4\}$ and

$B=\{4,5,6\}$, then

$A\cup B=\{1,2,3,4,5,6\}$ And then

$P(A\cup B)=P(\Omega)=1\cdot$

Or

$P(A)=\frac{4}{6}$, $P(A)=\frac{3}{6}$, $A\cap B=\{1\}$,

Then $P(A\cap B)=\frac{1}{6}$

Therefore,

$$\begin{aligned}P(A\cup B)&=P(A)+P(B)-P(A\cap B)\\&=\frac{4}{6}+\frac{3}{6}-\frac{1}{6}\\&=1\end{aligned}$$

Example 43

Events A and B are such that they are both mutually exclusive and exhaustive. Find the relation between these two events.

Solution

If A and B are mutually exclusive, then

$P(A \cup B) = P(A) + P(B)$

If A and B are exhaustive, then

$P(A \cup B) = 1$

Therefore,

$P(A) + P(B) = 1$

$P(B) = 1 - P(A)$

But, $P(A') = 1 - P(A)$

Therefore, $P(B) = P(A')$

i.e, $B = A'$

Similarly, $A = B'$

Thus, if events A and B are such that they are both mutually exclusive and exhaustive, then they are complementary.

Exercise 13

In a class of a certain school, there are 12 girls and 20 boys.
If a teacher wants to choose one learner to answer the asked question

a) What is the probability that the chosen learner is a girl?

b) What is the probability that the chosen learner is a boy?

c) If teacher doesn't care on the gender, what is the probability of choosing any learners?

Unit summary

1. Combinatorial analysis is a **mathematical theory of counting**.
2. **Experiment**: any human activity.

   Trial: small experiment contained in a large experiment.

   Outcome called **event**: a result of an experiment.
3. "If Experiments 1 through k have n_1 through n_k outcomes, respectively, then the experiment 1, 2, 3 … and k has $n_1 \times n_2 \times ... \times n_k$ outcomes."
4. The number of different permutations of n different objects (unlike objects) in a row is

   $$n \times (n-1) \times (n-2) \times (n-3) \times ... \times 2 \times 1$$
5. The number of different permutations of n objects with n_1 alike, n_2 alike, …, is $\dfrac{n!}{n_1!n_2!...}$.
6. The number of arrangements of n unlike things in a circle will therefore be $(n-1)!$. In those cases where clockwise and anticlockwise arrangements are not considered to be different, this reduces to $\frac{1}{2}(n-1)!$
7. "If Experiments 1 through k have respectively n_1 through n_k outcomes, then the experiment 1 or 2 or … or k has $n_1 + n_2 + ... + n_k$ outcomes.
8. The number of different permutations of r unlike objects selected from n different objects is ${}^nP_r = \dfrac{n!}{(n-r)!}$ or we can use the denotation $P_r^n = \dfrac{n!}{(n-r)!}$ or $P(n,r) = \dfrac{n!}{(n-r)!}$
9. The number of different groups of r items that could be formed from a set of n distinct objects with the order of selections being ignored is ${}^nC_r = \dfrac{n!}{(n-r)!r!}$.

10. For every integer $n \geq 1$, $(a+b)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r} b^r$

11. **Probability** is the chance that something will happen-how likely it is that some event will happen.

12. A **random experiment** is an experiment that, at least theoretically, may be repeated as often as we want and whose outcome cannot be predicted, the roll of a die.

13. Each time experiment is repeated, an **elementary outcome** is obtained.

14. The set of all elementary outcomes of a random experiment is called the **sample space**, which is denoted by Ω.

15. **Discrete sample space**: Firstly is the number of possible outcomes is **finite**.

16. **Continuous sample space**: If the sample space contains one or more intervals, the sample space is then **uncountable infinite**.

17. An **event** is a set of elementary outcomes. That is, it is a subset of the sample space.

18. Two or more events which have an equal probability of occurrence are said to be **equally likely**, i.e. if on taking into account all the conditions, there should be no reason to except any one of the events in preference over the others. Equally likely events may be simple or compound events.

19. Two events, A and B are said to be **incompatible** (or **mutually exclusive**) if their intersection is empty. We then write that $A \cap B = \varnothing$.

20. Two events, A and B are said to be **exhaustive** if they satisfy the condition $A \cup B = \Omega$.

21. An event which is sure to occur at every performance of an experiment is called a **certain event** connected with the experiment.

22. An event which cannot occur at any performance of the experiment is called an **impossible event.**

23. Two events are said to be **equivalent or identical** if one of them implies and implied by other. That is, the occurrence of one event implies the occurrence of the other and vice versa.
24. The outcomes which make necessary the happening of an event in a trial are called **favourable events**.
25. If E is an event, then E' is the event which occurs when E does not occur. Events E and E' are said to be **complementary events.**
26. The probability of an event $A \subset \Omega$, is a real number obtained by applying to A the function P defined by
$$P(A) = \frac{number\ of\ \ favourable\ outcomes}{number\ of\ \ possible\ outcomes} = \frac{\# A}{\# \rceil}$$
27. If A and B are events from a sample space E, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Revision exercise

1. If there are three different roads joining town A to town B and four different roads joining town B to town C, in how many different ways can I travel from A to C via B and return if:
   a) there are no restrictions,
   b) I am not able to return on any road I used on the outwards journey?
2. How many 3-digits numbers can be formed using only the digits 1, 2, 3, 5, 7 and 8? How many of these numbers are even? How many are less than 500?
3. There are three roads joining town X to town Y; three more roads join Y to z and two roads join Z to A. How many different routes are there from X to A passing through Y and Z?

4. In how many ways can a group of ten children be arranged in a line?
5. In how many ways can eight different books be arranged on a bookshelf?
6. The letters a, b, c and d are to be arranged in a row with each letter being used once and once only. In how many ways can this be done?
7. Evaluate:

   a) $\frac{15!}{13!}$ b) $\frac{9!}{11!}$ c) $6!-5!$ d) $\frac{10!-9!}{8!}$

8. Simplify:

   a) $\frac{(n-1)!}{(n-2)!}$ b) $\frac{(n+2)!}{n!}$ c) $\frac{(n+2)!-n!}{(n-1)!}$

9. There are ten teams in the local football competition. In how many ways can the first four places in the premiership table be filled?
10. In how many different ways can five identical blue balls, two identical red balls and a yellow ball be arranged in a row?
11. Find the number of arrangements of four different letters chosen from the word **PROBLEM** which:

    a) begin with a vowel.

    b) end with a consonant.

12. With his breakfast, a man sometimes has tea, sometimes coffee and sometimes fruit juice, but never more than one of these on any one day. Find the number of possible arrangements he can have in a period of 4 days if:

    a) he always has a drink of some sort,

    b) he may choose not to have a drink on certain days.

13. How many numbers greater than 300 can be formed from the figures 4, 3, 2 and 1 if each figure can be used not more than once in each number and all the figures need not be used each time?
14. A box contains 14 colored discs which are identical except for their color. There are 5 red discs, 4 green discs, 4 blue discs and 2 yellow. In how many ways can the 14 discs be arranged in a row?

15. Find the number of different three-letter arrangements that can be made from the letters of the word **ISOSCELES**.

    How many of these arrangements will contain:

    a) no **E**'s at all,
    b) atleast one **E**,
    c) no vowels at all,
    d) atleast one vowel?

16. A basketball team of 6 is to be chosen from 11 available players. In how many ways can this be done if:
    a) there no restrictions,
    b) 3 of the players are automatically selected,
    c) 3 of the players are automatic in selections and 2 other players are injured and cannot play?

17. In how many ways can 9 people be placed in cars which can take 2, 3 and 4 passengers respectively, assuming that the seating arrangements inside the cars are not important?

18. A group consists of 5 boys and 8 girls. In how many ways can a team of four be chosen, if the team contains:
    a) no girl.
    b) not more than one girl.
    c) atleast two boys?

19. Find the number of ways that 9 children can be divided into:
    a) a group of 5 and a group of 4 children,
    b) three groups of 3 children.

20.
    a) A small holiday hotel advertises for a manager and 7 other members of staff. There are 4 applicants for the position of manager and 10 other people apply for the other jobs at the hotel. Find the number of different ways of selecting a group of people for the 8 jobs.
    b) The hotel has 4 single rooms, 6 double rooms and 5 family rooms. For a particular week, 4 individuals book single rooms, 3 couples book double rooms and 3 families book family rooms. Given that all the rooms are available for that week, find the number of
    c) different possible arrangements of booking amongst the rooms.

d) One afternoon, 12 guests organize a game requiring 2 teams of 6. Find the number of different ways of selecting the teams.

e) Given that the 12 guests consist of 6 adults and 6 children and that each team must contain atleast 2 adults, find the number of different ways of selecting the teams.

21. Expand the following using binomial theorem:

a) $(3+x)^3$ b) $(5+2x)^3$ c) $(2+x)^4$ d) $(2-x)^4$

e) $(2y+x)^5$ f) $(2x-3y)^5$ g) $\left(x-\frac{1}{x}\right)^4$ h) $\left(x-\frac{2}{x}\right)^5$

22. Use the binomial theorem to expand $(1+x)^{12}$ in ascending powers of x up to and including the term in x^3.

23. Expand and simplify $\left(2x+\frac{1}{x^2}\right)^5+\left(2x-\frac{1}{x^2}\right)^5$.

24. Find the value of n if the coefficient of x^3 in the expansion of $(2+3x)^n$ is twice the coefficient of x^2.

25. The coefficient of x^5 in the expansion of $(1+5x)^8$ is equal to the coefficient of x^4 in the expansion of $(a+5x)^7$. Find the value of a.

26. Use the expansion of $(2-x)^5$ to evaluate $(1.98)^5$; correct to 5 decimal places.

27. Find the first four terms in the series expansion of $(a-3x)^{10}$ in ascending powers of x.

28. If x is such that terms involving x^5 and higher powers can be neglected, find an approximate expansion of $\left(1+\frac{x}{2}\right)^{20}$.

29. When $(1+ax)^n$ is expanded in ascending powers of x, the series expansion is $1+2x+\frac{15x^2}{8}+\ldots$; find the values of n and a.

30. Find the term independent of x in the expansion of each of the following:

a) $\left(3x+\frac{1}{x}\right)^{10}$ b) $\left(2x^2+\frac{4}{x}\right)^{12}$ c) $\left(\frac{3}{x^2}-2x\right)^{6}$ d) $\left(1+x^2\right)\left(2x+\frac{1}{x}\right)^{10}$

31. Use binomial expansion to evaluate the following to the stated degree of accuracy:

a) $(1.01)^9$ correct to four decimal places,

b) $(0.998)^7$ correct to seven decimal places,

c) $(0.99)^{10}$ correct to four decimal places,

d) $(1.99)^{10}$ correct to four significant figures.

32. Use binomial expansion to evaluate the following to the stated degree of accuracy:

a) $\sqrt{0.96}$ correct to five decimal places,

b) $\sqrt{104}$ correct to six significant figures,

c) $\sqrt{4.08}$ correct to four decimal places,

d) $\sqrt[4]{1.08}$ correct to five decimal places,

e) $\sqrt[3]{8.72}$ correct to five decimal places.

33. Expand $\sqrt{\frac{1-x}{1+2x}}$ in ascending power of x, up to and including the term in x^3. State the values of x for which the expansion is valid.

34. When the terms in x^4 and higher powers of x are neglected, the series expansion of $\frac{2+3x-x^2}{(1+2x)^3}$ in ascending powers of x gives $a+bx+cx^2+dx^3$. Find the values of a, b, c and d.

35. Find the coefficient of x^3 in the expansion of;

a) $(5+3x)^8$ b) $(7-2x)^7$

36. In the expansion of each of the following, find the coefficient of the specified power of x:

a) $(1+2x)^7$, x^4 b) $(5x-3)^7$, x^4 c) $(3x-2x^3)^5$, x^{11}

d) $\left(x-\frac{2}{x}\right)^8$, x^2 e) $\left(x^2+\frac{4}{x}\right)^{10}$, x^{-1} f) $\left(\frac{x}{6}-\frac{3}{x}\right)^9$, x^3

37. Find the possible values of the constant k if the coefficient of x^{-5} in the expansion of $\left(kx+\frac{1}{x}\right)^9$ is 16.

38. Find the term involving x^8 in the expansion of $(4+x)^{12}$

39. The first four terms in the expansion of $(1-x)^n$ are $1-6x+px^2+qx^3$. Show that $p=15$ and obtain the value of q.

40. Find the values of the constants a and b for which the expansions, in ascending powers of x, of the two expressions $(1+2x)^{\frac{1}{2}}$ and $\frac{1+ax}{1+bx}$ up to and including term in x^2, are the same. With these values of a and b, use the result $(1+2x)^{\frac{1}{2}} \approx \frac{1+ax}{1+bx}$, with $x=-\frac{1}{100}$, to obtain an approximate value for $\sqrt{2}$ in the form $\frac{p}{q}$, where p and q are positive integers.

41. Which of the following is an experiment?

a) Tossing a coin, b) Rolling a single die,

c) Choosing a marble from a jar, d) All of the above.

42. Which of the following is an outcome?

a) Rolling a pair of dice, b) Landing on red,

c) Choosing 2 marbles from a jar, d) None of the above.

43. Which of the following experiments does not have equally likely outcomes?

a) Choose a number at random between 1 and 7,

b) Toss a coin,

c) Choose a letter at random from the word SCHOOL,

d) None of the above.

44. A glass jar contains 5 red, 3 blue and 2 green jelly beans. If a jelly bean is chosen at random from the jar, then which of the following is an impossible event?
    a) Choosing a red jelly bean,
    b) Choosing a blue jelly bean,
    c) Choosing a yellow jelly bean,
    d) None of the above.
45. A spinner has 7 equal sectors numbered 1 to 7. If you spin the spinner, then which of the following is a certain event?
    a) Landing on a number less than 7,
    b) Landing on a number less than 8,
    c) Landing on a number greater than 1,
    d) None of the above.
46. Tickets numbered 1 to 20 are mixed up and then a ticket is drawn at random. What is the probability that the ticket drawn has a number which is a multiple of 3 or 5?
47. A bag contains 2 red, 3 green and 2 blue balls. Two balls are drawn at random. What is the probability that none of the balls drawn is blue?
48. In a box, there are 8 red, 7 blue and 6 green balls. One ball is picked up randomly. What is the probability that it is neither red nor green?
49. In a class, there are 15 boys and 10 girls. Three students are selected at random. What is the probability that 1 girl and 2 boys are selected?
50. From a pack of 52 cards, two cards are drawn together at random. What is the probability of both the cards being kings?
51. One card is drawn at random from a pack of 52 cards. What is the probability that the card drawn is a face card (Jack, Queen and King only)?
52. An integer is chosen at random from the first 200 positive integers. Find the probability that the number is:
    a) divisible by 2
    b) divisible by 7
    c) divisible by 2 and 7
    d) divisible by neither 2 nor 7
53. The letters of the word FACETIOUS are arranged in a row. Find the probability that:
    a) the first 2 letters are consonants,
    b) all the vowels are together.

References

A. J. Sadler, D. W. S. Thorning. *Understanding Pure Mathematics*. Oxford University Press 1987.

Arthur Adam Freddy Goossens and Francis Lousberg. *Mathématisons 65.* DeBoeck,3rd ed 1991.

Charles D. Hodgman, M.S. and Samuel M. Selby, Robert C.Weast. *Standard Mathematical Table.* Chemical Rubber Publishing Company, Cleveland, Ohio 1959.

David Rayner. *Higher GCSE Mathematics.* Oxford University Press 2000.

Direction des Progammes de l'Enseignement Secondaire. *Géometrie de l'Espace 1er Fascule.* Kigali, October1988.

Direction des Progammes de l'Enseignement Secondaire. *Géometrie de l'Espace 2ème Fascule.* Kigali, October1988.

Frank Ebos, Dennis Hamaguchi, Barbana Morrison & John Klassen, *Mathematics Principles & Process,* Nelson Canada A Division of International Thomson Limited 1990.

George B. Thomas, Maurice D. Weir & Joel R. Hass, *Thomas' Calculus Twelfth Edition,* Pearson Education, Inc. 2010.

Geoff Mannall & Michael Kenwood. *Pure Mathematics 2*, Heinemann Educational Publishers 1995.

H.K. Dass. *Engineering Mathematics*. New Delhi, S. Chand & Company Ltd, thirteenth revised edition 2007.

Hubert Carnec, Genevieve Haye, Monique Nouet, Rene Seroux, Jacqueline Venard. *Mathématiques TS Enseignement obligatoire.* Bordas Paris 1994.

James T. McClave, P.George Benson. *Statistics for Business and Economics.* USA, Dellen Publishing Company, a division of Macmillan, Inc 1988.

J. Crawshaw and J. Chambers. *A concise course in A-Level statistics with worked examples*, Stanley Thornes (Publishers) Ltd, 1984.

Jean Paul Beltramonde, Vincent Brun, Claude Felloneau, Lydia Misset, Claude Talamoni. *Declic 1re S Mathématiques.* Hachette-education, Paris 2005.

J. F. Talber & H. H. Heing. *Additional Mathematics 6th Edition Pure & Applied.* Pearson Education South Asia Pte Ltd 1995.

J.K. Backhouse, S. P. T. Houldsworth B.E.D. Copper and P.J.F. Horril. *Pure Mathematics 2.* Longman, 3rd ed 1985, fifteenth impression 1998.

Mukasonga Solange. *Mathématiques 12, AnalyseNumérique.* KIE, Kigali2006.

N. Piskounov. *Calcul Différential et Integral tom II 9ème édition.* Editions MIR. Moscou, 1980.

Paule Faure-Benjamin Bouchon. *Mathématiques Terminales F.* Editions Nathan, Paris 1992.

Peter Smythe. *Mathematics HL & SL with HL options, Revised Edition*, Mathematics Publishing Pty. Limited, 2005.

Robert A. Adms & Christopher Essex. *Calculus A complete Course, 7th ed.* Pearson Canada Inc., Toronto, Ontario 2010.

Seymour Lipschutz. *Schaum's Outline of Theory and Problems of Linear Algebra.* McGraw-Hill 1968.

Shampiyona Aimable. *Mathématiques 6.* Kigali, Juin 2005.

Yves Noirot, Jean-Paul Parisot, Nathalie Brouillet. *Cours de Physique Mathématiques pour la Physique*. Paris, Dunod, 1997.